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An MLE approach for estimating the time of step changes in Poisson regression profiles

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Abstract In some process control applications, the quality of a product or process can be characterized by a relationship between two or more variables, which is typically referred to as a profile. Moreover, in some situations, the dependent variable is a count, which can be modeled as a Poisson regression of one explanatory variable. We refer to this as Poisson regression profiles. Control chart signals do not indicate the real time of process changes, so estimators are applied to indicate the time when a change in the process takes place, which is referred to as the change point. In this paper, we propose the use of an MLE estimator to identify the real time of a step change in phase II monitoring of Poisson regression profiles. The results reveal that the change point estimator is effective in identifying step shifts in the process parameters.

1. Introduction

In some process control applications, quality of a product or process can be represented by a relationship between a response variable and one or more independent variables, instead of a single quality characteristic or a multivariate quality vector. This relationship is usually referred to as a profile. Some of the practical applications of profile monitoring have been reported by researchers, including Mestek et al. [1], Kang and Albin [2], Mahmoud and Woodall [3] and Amiri et al. [4]. Different methods have been developed to monitor different types of profiles in both phases I and II. In phase I, one evaluates the process stability and estimates its parameters based on a historical data set. However, the purpose of phase II analysis is to detect shifts in the process parameters as soon as possible. Many authors, such as Kang and Albin [2], Kim et al. [5], Mahmoud et al. [6] Zou et al. [7], Saghaei et al. [8] and Zhang et al. [9], have investigated phase I and phase II monitoring of simple linear profiles. Some methods are proposed to monitor more complicated models, such as multiple linear regression, polynomial and nonlinear profiles. (See, for example [10–12].)

All of the aforementioned researchers assume that the response variable is continuous (usually Normal) and characterize profiles with linear or nonlinear models. However, in many industrial applications, the response variable is discrete, such as binary or countable, which is often modeled by a generalized linear model. However, profile monitoring, when the response is binary or a count, has received very little attention in the literature. Yeh et al. [13] studied binary profiles in phase I. They proposed different T^2 control charts for monitoring logistic regression profiles. Shang et al. [14] proposed a control scheme based on EWMA–GLM to represent the relationship between binary response and random explanatory variables in phase II. Their approach assumes that explanatory variables are random and different in each sample.

Control charts have proven to be very effective in detecting out of control signals. When a control chart signals a change in the process parameter, knowing when a process has changed, which is referred to as the change point, would simplify the search for and identification of the special cause. Consequently, having an estimate of the process change point would be useful to process analysts. Further, it could reduce the risk of

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misdiagnosing the control chart signal, which often leads to unnecessary and costly adjustments to the process. Change point problems are classified according to change types including step, drift and monotonic shifts. To find the real time of a change, many authors have suggested several methods, such as Maximum Likelihood Estimator (MLE), cumulative sum (CUSUM), Exponentially Weighted Moving Average (EWMA) and intelligent methods (artificial network, clustering and decision tree). Accordingly, Samuel et al. [15,16] proposed maximum likelihood estimators for the time of step changes in the mean and variance of a normal distribution, respectively. Pignatiello and Samuel [17,18] proposed an MLE in different control charts to find the real time of a change point under a step shift. Perry and Pignatiello [19,20] considered a linear trend change in the mean of the Poisson and normal processes, respectively. Pignatiello and Samuel [18] and Perry and Pignatiello [19] showed that the performance of the MLE is better than the estimators of EWMA and CUSUM in identifying the change point of a normal and Poisson process, respectively. Amiri and Khosravi [21] proposed an MLE change point estimator in high quality processes under a drift in proportion to nonconforming items when a signal is triggered by a cumulative count of conformance control charts.

Many authors have studied the change point problem in regression models, but under a different sampling framework from that of the profile data. These authors assumed that either there is a possible change point after any single observation or that data are obtained sequentially, one observation at a time. The change point problem in a regression model is usually referred to as segmented regression. In the profile applications, multiple data sets are collected over time in a functional data sampling framework. Mahmoud et al. [6] and Zou et al. [7] proposed methods based on likelihood ratio statistics to estimate the step change point in simple linear profiles in phases I and II, respectively. Kazemzadeh et al. [11] used the same method to estimate the change point in polynomial profiles under a step shift in phase I. Sharafi et al. [22] suggested an MLE method to identify the real time of a step change in phase II monitoring of binary profiles. Nevertheless, to the best of our knowledge, there is no method for estimating the real time of a step change in Poisson regression profiles in both phases I and II. (See a comprehensive review on change point estimation methods for control chart post signal diagnostics by Amiri and Allahyari [23]). Therefore, in this paper we propose an MLE method to estimate step shift in phase II monitoring of Poisson regression profiles. The rest of this paper is organized as follows. Section 2 illustrates the Poisson regression model and explains the steps of estimating the model parameters. Section 3 presents the change point model and assumptions of the problem. The performance of the proposed model is investigated in Section 4. Conclusions and some future research ideas are provided in the final section.

2. Poisson regression model

Poisson distribution is often used to model information on counts of various kinds, particularly in situations where there is no natural denominator, and thus, no upper bound or limit on how large an observed count can be. This is in contrast to the binomial distribution, which focuses on the observed proportions. Possible examples of count data, where a Poisson model is useful, include:

(i) The number of automobile fatalities in a given region over yearly intervals,
(ii) The number of AIDS cases for a given risk group for a series of monthly intervals,
(iii) The number of murders in a city by year,
(iv) The number of server failures for a web-based company by year.

The Poisson regression model is a useful tool for the analysis of these data. This model belongs to a class of models called generalized linear models. In a Generalized Linear Model (GLM), the mean of the response, $\boldsymbol{\mu}$, is modeled as a monotonic transformation of a linear function of the explanatory variables, $g(\beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_p x_p)$. In this function, there are p predictor variables for any of n independent experimental sets, which are shown by $\mathbf{x}_i = (x_{i1}, x_{i2}, \ldots, x_{ip})^T$ and $i = 1, 2, \ldots, n$. The Poisson regression model assumes that the dependent variable of observation, y_i, is modeled as a Poisson random variable with mean λ_i, and each λ_i is a function of $\mathbf{x}_i$, $i = 1, 2, \ldots, n$. Note that the equal variance assumption of classic linear regression is violated, because the y_i have means equal to their variances. In Poisson regression, this function is the log function, so the model is set as:

$$g(\lambda_i) = \log(\lambda_i) = \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_p x_{ip}, \tag{1}$$

where:

$$\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_p)^T,$$

is the regression parameter vector. We say that the Poisson regression model is a generalized linear model with Poisson error and a log link function. It is usual to set $x_{i1} \equiv 1$ in order for β_i to be the intercept of the model. The alternative Eq. (1), directly specifying λ_i, is:

$$\lambda_i = \exp(\mathbf{x}_i^T \boldsymbol{\beta}) = \exp(\eta_i), \tag{2}$$

where:

$$\eta_i = \mathbf{x}_i^T \boldsymbol{\beta} = \sum_{k=1}^{p} \beta_k x_{ik},$$

is the usual linear combination of predictors for the level of i. The response variable is y_i, which follows a Poisson distribution with parameter λ_i. The parameters of Eq. (2) can be estimated by the maximum likelihood method. Albert and Anderson [24] used the following likelihood function to estimate the model parameters:

$$L(\boldsymbol{\lambda}, \mathbf{y}) = \prod_{i=1}^{n} \left[e^{-\lambda_i} \frac{(\lambda_i)^{y_i}}{y_i!} \right] = e^{-\sum_{i=1}^{n} \lambda_i} \frac{\prod_{i=1}^{n} (\lambda_i)^{y_i}}{\prod_{i=1}^{n} y_i!}, \tag{3}$$

where:

$$\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)^T,$$

and:

$$\mathbf{y} = (y_1, y_2, \ldots, y_n)^T.$$

Taking the logarithm of Eq. (3) and using $\eta_i = \mathbf{x}_i^T \boldsymbol{\beta} = \sum_{k=1}^{p} \beta_k x_{ik} = \log \lambda_i$, one can reexpress the log-likelihood as:

$$l(\boldsymbol{\lambda}, \mathbf{y}) = -\sum_{i=1}^{n} \exp(\mathbf{x}_i^T \boldsymbol{\beta}) + \sum_{i=1}^{n} y_i \ln(\exp(\mathbf{x}_i^T \boldsymbol{\beta})) - \sum_{i=1}^{n} \ln(y_i!). \tag{4}$$

Taking the derivative of Eq. (4), with respect to $\boldsymbol{\beta}$, and using the iterative weighted least square estimation method suggested

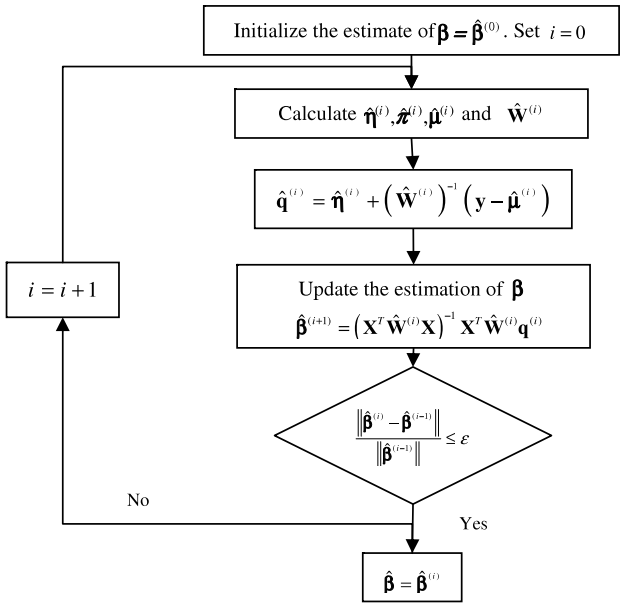

Figure 1: The procedure of estimation of Poisson regression parameters.

by McCullagh and Nelder [25], the Poisson regression parameters can be estimated as follows:

$$\widehat{\boldsymbol{\beta}} = \left(\mathbf{X}^T\hat{\mathbf{W}}\mathbf{X}\right)^{-1}\mathbf{X}^T\hat{\mathbf{W}}\mathbf{q}. \tag{5}$$

In Eq. (5), $\mathbf{X} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)^T$ is an $n \times p$ matrix, $\hat{\mathbf{W}} = \text{diag}[\hat{\lambda}_1, \hat{\lambda}_2, \ldots, \hat{\lambda}_n]$ is an $n \times n$ diagonal matrix and $\mathbf{q} = \widehat{\boldsymbol{\beta}} + \hat{\mathbf{W}}^{-1}(\mathbf{y} - \hat{\boldsymbol{\mu}})$. The procedure iterations are described in Figure 1.

McCullagh and Nelder [25] proved that as n becomes large, $\hat{\boldsymbol{\beta}}$ is distributed asymptotically as a p-dimensional normal distribution, $N_p(\boldsymbol{\beta}, (\mathbf{X}^T\mathbf{W}\mathbf{X})^{-1})$, where:

$$\mathbf{S} = (\mathbf{X}^T\mathbf{W}\mathbf{X})^{-1} = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}.$$

This procedure will be used in the MLE change point estimator described in Section 3.

3. MLE change point estimator

Here, it is assumed that the underlying process initially operates in a state of statistical control, with observations coming from a Poisson distribution with the known parameters $\boldsymbol{\beta} = \boldsymbol{\beta}_0$ ($\boldsymbol{\beta}$ is a p-dimensional vector); so, the mass probability function is:

$$f\left(\mathbf{y}_{ij}\right) = e^{-\lambda_i}\frac{(\lambda_i)^{y_{ij}}}{y_{ij}!} = e^{-\exp(\mathbf{x}_i\boldsymbol{\beta}_0)}\frac{(\exp(\mathbf{x}_i\boldsymbol{\beta}_0))^{y_{ij}}}{y_{ij}!},$$

where y_{ij} is the value taken by the response variable for the ith value of the predictor variable in the jth profile. After an unknown amount of time elapses, in an unknown profile, τ (known as the process change point), the process changes to an unknown out-of-control state, such that the behavior in $\boldsymbol{\beta}$ can be described by $\boldsymbol{\beta}_1 = \boldsymbol{\beta}_0 + \boldsymbol{\Delta}$, and it remains at the new level until the source of the assignable cause is identified and eliminated, where $\boldsymbol{\Delta} = (\delta_1\sigma_1, \delta_2\sigma_2)^T$ and δ_1, δ_2 are constant. On the other hand, during the formulation of profiles, $j = 1, 2, \ldots, \tau$, the process parameter, $\boldsymbol{\beta}$, is equal to its known in-control value, $\boldsymbol{\beta}_0$. For profiles $j = \tau + 1, \tau + 2, \ldots, T$, parameter $\boldsymbol{\beta}$ becomes equal to some unknown parameter, $\boldsymbol{\beta}_1$, where T is the last profile sampled, in which the control chart signaled an out-of-control state. Here, we describe the level of shifts in $\boldsymbol{\beta}$, based on the non-centrality parameter (ncp), which is defined as ncp $= \boldsymbol{\Delta}^T\mathbf{S}^{-1}\boldsymbol{\Delta}$. Two unknown parameters in the model are τ and $\boldsymbol{\beta}_1$, representing the last profile taken from an in-control process and the out-of-control process parameter, respectively. To estimate these unknown parameters, this paper uses an MLE approach, and the proposed change-point estimator is denoted as $\hat{\tau}$. Assuming a process change point at τ, the likelihood function is given by:

$$L(\tau, \boldsymbol{\beta}_1|\mathbf{y}) = \frac{1}{\prod_{j=1}^{T}\prod_{i=1}^{n}\mathbf{y}_{ij}!}\left[e^{-\sum_{j=1}^{\tau}\sum_{i=1}^{n}\exp(\mathbf{x}_i\boldsymbol{\beta}_0) - \sum_{j=\tau+1}^{T}\sum_{i=1}^{n}\exp(\mathbf{x}_i\boldsymbol{\beta}_1)} \times \exp(\mathbf{x}_i\boldsymbol{\beta}_0)^{\sum_{j=1}^{\tau}\sum_{i=1}^{n}y_{ij}}\exp(\mathbf{x}_i\boldsymbol{\beta}_1)^{\sum_{j=\tau+1}^{T}\sum_{i=1}^{n}y_{ij}}\right]. \tag{6}$$

The MLE of τ is the value of τ that maximizes the likelihood function in Eq. (6) or, equivalently, its logarithm. Taking the logarithm of Eq. (6), we have:

$$\ln(\tau, \beta_1|\mathbf{y}) = -\sum_{j=1}^{\tau}\sum_{i=1}^{n}\exp(\mathbf{x}_i\boldsymbol{\beta}_0) - \sum_{j=\tau+1}^{T}\sum_{i=1}^{n}\exp(\mathbf{x}_i\boldsymbol{\beta}_1) + \sum_{j=1}^{\tau}\sum_{i=1}^{n}y_{ij}\mathbf{x}_i\boldsymbol{\beta}_0 + \sum_{j=\tau+1}^{T}\sum_{i=1}^{n}y_{ij}\mathbf{x}_i\boldsymbol{\beta}_1. \tag{7}$$

To determine the unknown parameters in Eq. (7) (τ and $\boldsymbol{\beta}_1$), an expression is required for these parameters, which maximize the log-likelihood function in Eq. (7), denoting $\hat{\tau}$ as $\hat{\boldsymbol{\beta}}_1$. We should take the partial derivative of Eq. (7), with respect to $\boldsymbol{\beta}_1$, and solve it to find the MLE of the parameter, and denote it as $\hat{\boldsymbol{\beta}}_1$.

$$\frac{\partial \ln L(\tau, \boldsymbol{\beta}_1|\mathbf{y})}{\partial \boldsymbol{\beta}_1} = -\sum_{j=\tau+1}^{T}\sum_{i=1}^{n}\mathbf{x}_i e^{\mathbf{x}_i\boldsymbol{\beta}_1} + \sum_{j=\tau+1}^{T}\sum_{i=1}^{n}y_{ij}\mathbf{x}_i = 0. \tag{8}$$

Since vector $\boldsymbol{\beta}$ appears with X in the terms of Eq. (7), an expression of $\mathbf{x}_i\boldsymbol{\beta}_1$ can also solve this problem. For a fixed value for τ, the MLE for $\mathbf{x}_i\boldsymbol{\beta}_1$ is:

$$\mathbf{x}_i\hat{\boldsymbol{\beta}}_1 = \ln\left[\sum_{j=\tau+1}^{T}\sum_{i=1}^{n}y_{ij}\mathbf{x}_i \Big/ (T - \tau)\right]. \tag{9}$$

By obtaining $\mathbf{x}_i\hat{\boldsymbol{\beta}}_1$, replacing it in Eq. (7) and calculating the logarithm of the likelihood function in Eq. (7) for all possible change-point values, the MLE of the change point, τ, is the value which maximize the expression in Eq. (7). The resulting estimate of the change point is as follows:

$$\hat{\tau} = \arg\max\left[-\sum_{j=1}^{\tau}\sum_{i=1}^{n}\exp(\mathbf{x}_i\boldsymbol{\beta}_0) - \frac{1}{T-\tau}\sum_{j=\tau+1}^{T}\sum_{i=1}^{n}y_{ij} + \sum_{j=1}^{\tau}\sum_{i=1}^{n}y_{ij}\mathbf{x}_i\boldsymbol{\beta}_0 + \sum_{j=\tau+1}^{T}\sum_{i=1}^{n}y_{ij} \times \ln\left[\sum_{j=\tau+1}^{T}\sum_{i=1}^{n}y_{ij}\mathbf{x}_i \Big/ (T-\tau)\right]\right], \tag{10}$$

where $\hat{\tau}$ is the MLE of the change point.

Table 1: Expected number of samples until the signal, the average of the change point estimate and standard deviations of the change point estimator average with 10,000 simulations runs when $p = 2$ and $\tau = 50$.

ncp	(δ_1, δ_2)	$E(T)$	$\bar{\hat{\tau}}$	$se(\hat{\tau})$
2.08	(0.1, 0.1)	149.4	50.72	2.15
3.70	(0.2, 0.2)	80.09	50.52	1.27
3.72	(0.1, 0.3)	79.02	50.44	1.02
5.78	(0.3, 0.2)	61.01	49.67	0.73
5.80	(0.4, 0.1)	61.10	50.25	0.69
5.85	(0.0.5)	59.78	50.22	0.79
8.32	(0.3, 0.3)	55.10	49.82	0.61
8.34	(0.2, 0.4)	55.12	50.07	0.59
11.33	(0.35, 0.35)	52.63	50.08	0.63
14.80	(0.4, 0.4)	51.69	50.02	0.25
14.81	(0.3, 0.5)	51.71	49.98	0.24
23.13	(0.5, 0.5)	51.12	49.94	0.21
23.38	(1, 0)	51.11	49.98	0.16
23.38	(0, 1)	51.10	49.95	0.19
33.31	(0.6, 0.6)	51.01	49.96	0.19

In this paper, we used a shewhart T^2 control chart to monitor a Poisson regression profile in phase II. Note that Yeh et al. [13] introduced five Hotelling T^2 control charts to monitor binary profiles in phase I. Any of these T^2 charts uses a different method to estimate the mean vector and covariance matrix. They showed that the T^2 control chart, which estimates the covariance matrix by averaging the covariance estimates of each given sample, is more effective in detecting both step and drift shifts. This control chart is applied in phase II with the assumption that the mean vector and covariance matrix are known. The T^2 statistic for sample j, $j = 1, 2, \ldots, T$ in phase II is defined as:

$$T_j^2 = \left(\hat{\boldsymbol{\beta}}_j - \boldsymbol{\beta}_0\right)^T \boldsymbol{\Sigma}^{-1} \left(\hat{\boldsymbol{\beta}}_j - \boldsymbol{\beta}_0\right), \tag{11}$$

where $\boldsymbol{\beta}_0$ and $\boldsymbol{\Sigma}$ are the mean vector and covariance matrix of Poisson regression parameters, respectively, when the process is in control.

The upper control limit for the proposed control chart is equal to $\chi^2_{2,\alpha}$, which is the α percentile point of the chi-square distribution with 2 degrees of freedom. The covariance matrix in phase II is also computed using the following equation:

$$\boldsymbol{\Sigma} = \left(\mathbf{X}^T\mathbf{W}\mathbf{X}\right)^{-1}. \tag{12}$$

Whenever the T^2 control chart signals an out-of-control state, the real time of a change can be estimated via Eq. (10).

4. Performance of the MLE estimator

In this section, the performance of the proposed estimator is examined using Monte Carlo simulation through two examples. In the first example, the number of explanatory variables is equal to 2. However, 3 explanatory variables are considered in Example 2.

4.1. Example 1

In this example, the number of explanatory variables is equal to 2 ($p = 2$). Thus, the link function is simplified as $g(\pi_i) = \beta_1 + \beta_2 x_i$, where β_1 and β_1 are the intercept and the slope of the regression function, respectively, which is shown by vector $\boldsymbol{\beta} = (\beta_1, \beta_2)^T$. Also, we set the design matrix $\mathbf{X}$ as:

$$\mathbf{X} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \log(1) & \log(2) & \cdots & \log(9) \end{pmatrix}^T.$$

It is assumed that the in-control $\boldsymbol{\beta}$ is $\boldsymbol{\beta}_0(1, 1.5)^T$, which comes from the historical dataset in phase I. The covariance matrix of the Poisson regression parameters ($\boldsymbol{\Sigma}$) in phase II is computed by Eq. (12) as follows:

$$\boldsymbol{\Sigma} = \left(\mathbf{X}^T\mathbf{W}\mathbf{X}\right)^{-1} = \begin{pmatrix} 0.07787 & -0.04022 \\ -0.04022 & 0.02170 \end{pmatrix}.$$

The upper control limit for the T_I^2 control chart is equal to $\chi^2_{2,0.005} = 10.59$.

Now, suppose an out-of-control process whose parameter vector $\boldsymbol{\beta}$ shifts from $\boldsymbol{\beta}_0$ to $\boldsymbol{\beta}_1 = \boldsymbol{\beta}_0 + \boldsymbol{\Delta}$, where $\boldsymbol{\Delta} = (\delta_1\sigma_1, \delta_2\sigma_2)^T$ and δ_1, δ_2 are constant.

A Monte Carlo simulation study is accomplished to examine the performance of the estimator. In this study, the process change point is considered at $\tau = 50$. During the formation of profiles $j = 1, 2, \ldots, 50$, the process parameter is equal to its known in-control value of $\boldsymbol{\beta}_0$. Therefore, for these profiles, the dependent observations are randomly generated from a Poisson regression with parameter vector $\boldsymbol{\beta}_0 = (1, 1.5)^T$. Starting at profile 51, observations are simulated from the out-of-control process with $\boldsymbol{\beta}_1$ until the T^2 control chart signals an out-of-control state. At this time, the change point estimator in Eq. (10) is used and the real time of the process change is determined. This procedure is repeated 10,000 times for different step shifts considered in the paper.

The simulation results are demonstrated in Tables 1 and 2. Table 1 shows the expected length of each simulation run $E(T)$,

Table 2: Estimated precision performances over a range of ncp values with 10,000 simulations runs when $p = 2$ and $\tau = 50$.

ncp	(δ_1, δ_2)	$\hat{p}(\lvert\hat{\tau}-\tau\rvert = 0)$	$\hat{p}(\lvert\hat{\tau}-\tau\rvert \le 1)$	$\hat{p}(\lvert\hat{\tau}-\tau\rvert \le 2)$	$\hat{p}(\lvert\hat{\tau}-\tau\rvert \le 3)$	$\hat{p}(\lvert\hat{\tau}-\tau\rvert \le 4)$	$\hat{p}(\lvert\hat{\tau}-\tau\rvert \le 5)$	$\hat{p}(\lvert\hat{\tau}-\tau\rvert \le 6)$
2.08	(0.1, 0.1)	0.60	0.77	0.85	0.92	0.95	0.96	0.97
3.70	(0.2, 0.2)	0.71	0.88	0.93	0.96	0.98	0.99	1.00
3.72	(0.1, 0.3)	0.73	0.89	0.95	0.98	0.99	1.00	
5.78	(0.3, 0.2)	0.83	0.94	0.98	0.99	1.00		
5.80	(0.4, 0.1)	0.79	0.94	0.98	0.99	0.99		
5.85	(0.0.5)	0.80	0.94	0.97	0.98	0.99		
8.32	(0.3, 0.3)	0.87	0.97	0.99	0.99	1.00		
8.34	(0.2, 0.4)	0.87	0.98	0.99	1.00			
11.33	(0.35, 0.35)	0.91	0.98	0.99	1.00			
14.80	(0.4, 0.4)	0.94	0.99	1.00				
14.81	(0.3, 0.5)	0.94	0.99	1.00				
23.13	(0.5, 0.5)	0.95	0.99	1.00				
23.38	(1, 0)	0.97	0.99	1.00				
23.38	(0, 1)	0.96	0.99	1.00				
33.31	(0.6, 0.6)	0.97	1.00					

Table 3: Summary of process parameters and specifications of Example 2.

Number of predictor variables	$p = 3$
Matrix of explanatory variables	$\mathbf{X} = \begin{pmatrix} 1 & \log(10) & \log(2500) \\ 1 & \log(15) & \log(2800) \\ 1 & \log(20) & \log(3100) \\ \vdots & \vdots & \vdots \\ 1 & \log(50) & \log(4900) \end{pmatrix}$
The in control Poisson regression parameter	$\boldsymbol{\beta}_0 = (-4, 2, 1)^T$
The upper control limit for the control chart	$\chi^2_{2,0.005} = 10.59$
The covariance matrix of the Poisson regression parameters in phase II	$S_I = \begin{pmatrix} 0.226470 & 0.016846 & -0.034461 \\ 0.016846 & 0.001280 & -0.002575 \\ -0.034461 & -0.002575 & 0.005248 \end{pmatrix}$
Process change point	$\tau = 50$

Table 4: Expected number of samples until the signal, the average of the change point estimate and standard deviations of the change point estimator average with 10,000 simulations runs when $p = 3$ and $\tau = 50$.

ncp	$(\delta_1, \delta_2, \delta_3)$	$E(T)$	$\bar{\hat{\tau}}$	$se(\hat{\tau})$
0.464	(0, 0.01, 0)	161.67	49.36	6.23
0.485	(0.001, 0.002.0.001)	161.81	49.40	6.44
0.936	(0, 0.005.0.002)	126.26	49.66	2.83
1.42	(0.003, 0.002.0.001)	101.74	49.73	2.56
2.13	(0.002, 0.005.0.002)	83.19	49.81	2.43
3.06	(0.003, 0.001.0.003)	67.74	49.82	2.4
4.74	(0.005, 0..0.003)	57.09	49.90	1.67
5.35	(0.005, 0.002.0.003)	55.51	49.92	1.17
6.09	(0.01, 0.0)	54.04	49.94	0.66
7.88	(0.010, 0.005.0)	53.12	49.94	0.58
9.05	(0.006, 0.004.0.004)	51.95	49.95	0.55
9.86	(0, 0.0.01)	51.63	49.96	0.47
11.49	(0.005, 0.004.0.006)	51.36	49.96	0.25
14.23	(0.006, 0.006.0.006)	51.14	49.97	0.23
18.20	(0.010, 0.008.0.004)	51.08	49.98	0.11
25.30	(0.008, 0.008.0.008)	51.04	49.99	0.09

which is the expected value of the number of samples taken until the first alarm is given by the control chart, i.e. $E(T) = \text{ARL} + 50$. Ideally, this statistic should be 51, since the change occurs after the 50th observation. Table 1 also shows the average change point estimate and the standard deviation of the change point estimator average under different magnitudes of step shifts considered. If the estimate of τ is perfect, the value of $\bar{\hat{\tau}}$ will be 50, indicating the last observation before the change occurs in the process.

From Table 1, we conclude that for shifts equal to 2.08, the expected number of samples are taken until the signal is 149.4. For this case, the average of the change point estimate is 50.72, which is quite close to the actual change point of $\tau = 50$. Moreover, the standard deviation of the change point estimator average is 2.15. Hence, our proposed change point estimator works satisfactorily, even under a small magnitude of shifts. Furthermore, as the magnitude of the step change increases, the performance of the estimator improves significantly.

Table 2 shows the results of the proportion of 10,000 simulation runs, showing that the estimator lies within a specified tolerance of the real change point value. The results provided in Table 2 are similar to those provided in Table 1. For example, if ncp $= 2.08$, the estimated probability that $\hat{\tau}$ lies within 1 or less from the real change point is 0.77. Also in this case, in 60% of the simulation runs, the estimator correctly identifies the real time of the change. From Table 2, we notice that the percentage of those simulation trials identifying the change point correctly are 60%, 71%, 83%, 87%, 91% and 94%, for the magnitude of shifts, ncp $= 2.08$, 3.70, 5.78, 8.32, 11.33 and 14.80, respectively. The probability of determining the change point within m observations from the actual time of the change increases as the magnitude of the step shift, ncp, increases.

4.2. Example 2

As mentioned before, in a generalized linear model, the mean of the response variable is modeled as a linear function of the explanatory variables, $\beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_p x_p$. As

Table 5: Estimated precision performances over a range of ncp values with 10,000 simulation runs when $p = 3$ and $\tau = 50$.

ncp	$(\delta_1, \delta_2, \delta_3)$	$\hat{p}(\lvert\hat{\tau}-\tau\rvert = 0)$	$\hat{p}(\lvert\hat{\tau}-\tau\rvert \le 1)$	$\hat{p}(\lvert\hat{\tau}-\tau\rvert \le 2)$	$\hat{p}(\lvert\hat{\tau}-\tau\rvert \le 3)$	$\hat{p}(\lvert\hat{\tau}-\tau\rvert \le 4)$	$\hat{p}(\lvert\hat{\tau}-\tau\rvert \le 5)$	$\hat{p}(\lvert\hat{\tau}-\tau\rvert \le 6)$
0.464	(0, 0.01, 0)	0.30	0.51	0.65	0.71	0.74	0.80	0.85
0.485	(0.001, 0.002.0.001)	0.30	0.51	0.66	0.73	0.79	0.84	0.87
0.936	(0, 0.005.0.002)	0.46	0.70	0.81	0.88	0.92	0.94	0.95
1.42	(0.003, 0.002.0.001)	0.61	0.83	0.93	0.96	0.97	0.97	0.97
2.13	(0.002, 0.005.0.002)	0.72	0.89	0.95	0.97	0.98	0.99	0.99
3.06	(0.003, 0.001.0.003)	0.78	0.93	0.97	0.98	0.99	0.99	1.00
4.74	(0.005, 0..0.003)	0.89	0.96	0.97	0.98	0.98	0.99	1.00
5.35	(0.005, 0.002.0.003)	0.92	0.97	0.98	0.99	1.00		
6.09	(0.01, 0.0)	0.92	0.98	0.99	0.99	1.00		
7.88	(0.010, 0.005.0)	0.96	0.98	0.99	0.99	1.00		
9.05	(0.006, 0.004.0.004)	0.96	0.98	0.99	1.00			
9.86	(0, 0.0.01)	0.97	0.99	0.99	1.00			
11.49	(0.005, 0.004.0.006)	0.98	0.99	0.99	1.00			
14.23	(0.006, 0.006.0.006)	0.98	0.99	1.00				
18.20	(0.010, 0.008.0.004)	0.99	1.00					
25.30	(0.008, 0.008.0.008)	0.99	1.00					

a second example, we set another type of Poisson regression model under $p = 3$ predictor variables for any of n independent experimental sets. The procedure applied for this example is exactly similar to Example 1. The explanatory variables, the regression parameters, and the mean and covariance matrix of the regression parameters are completely defined in Table 3.

The results of simulation studies are given in Tables 4 and 5. Table 4 shows the accuracy performance of the change point estimator, and Table 5 shows the results of the proportion of 10,000 simulation runs, showing that the estimator lies within a specified tolerance of the real change point value. Similar results are obtained in this example, which shows the acceptable performance of the proposed change point estimator.

5. Conclusions

In this paper, an MLE approach is proposed for identifying the time of a step change in phase II monitoring of Poisson regression profiles, when the type of change is step shift. The performance of the proposed change point estimator is evaluated through simulation studies. The results show that the change point estimator performs satisfactorily in all shifts considered in this paper. For future research, one can develop an MLE change point estimator for other distributions of the exponential family, such as logistic and Gamma. In addition, other types of change, including drift or isotonic changes, could be investigated by researchers.

References

[1] Mestek, O., Pavlik, J. and Suchnek, M. "Multivariate control charts: control charts for calibration curves", *Fresenius' Journal of Analytical Chemistry*, 350, pp. 344–351 (1994).
[2] Kang, L. and Albin, S.L. "Online monitoring when the process yields a linear profile", *Journal of Quality Technology*, 32, pp. 418–426 (2000).
[3] Mahmoud, M.A. and Woodall, W.H. "Phase I analysis of linear profiles with calibration applications", *Technometrics*, 46, pp. 380–391 (2004).
[4] Amiri, A., Jensen, W.A. and Kazemzadeh, R.B. "A case study on monitoring polynomial profiles in the automotive industry", *Quality and Reliability Engineering International*, 26, pp. 509–520 (2010).
[5] Kim, K., Mahmoud, M.A. and Woodall, W.H. "On the monitoring of linear profiles", *Journal of Quality Technology*, 35, pp. 317–328 (2003).
[6] Mahmoud, M.A., Parker, P.A., Woodall, W.H. and Hawkins, D.M. "A change point method for linear profile data", *Quality and Reliability Engineering International*, 23, pp. 247–268 (2007).
[7] Zou, C., Zhang, Y. and Wang, Z. "Control chart based on change point model for monitoring linear profiles", *IIE Transactions*, 38, pp. 1093–1103 (2006).
[8] Saghaei, A., Mehrjoo, M. and Amiri, A. "A CUSUM-based method for monitoring simple linear profiles", *The International Journal of Advanced Manufacturing Technology*, 45, pp. 1252–1260 (2009).
[9] Zhang, J., Li, Z. and Wang, Z. "Control chart based on likelihood ratio for monitoring linear profiles", *Computational Statistics and Data Analysis*, 53, pp. 1440–1448 (2009).
[10] Amiri, A., Eyvazian, M., Zou, C. and Noorossana, R. "A parameters reduction method for monitoring multiple linear regression profiles", *The International Journal of Advanced Manufacturing Technology*, 58, pp. 621–629 (2012).
[11] Kazemzadeh, R.B., Noorossana, R. and Amiri, A. "Phase I monitoring of polynomial profiles", *Communications in Statistics—Theory and Methods*, 37, pp. 1671–1686 (2008).
[12] Williams, J.D., Woodall, W.H. and Birch, J.B. "Statistical monitoring of nonlinear product and process quality profiles", *Quality and Reliability Engineering International*, 23, pp. 925–941 (2007).
[13] Yeh, B., Huwang, L. and Li, Y.M. "Profile monitoring for a binary response", *IIE Transactions*, 41, pp. 931–941 (2009).
[14] Shang, Y., Tsung, F. and Zou, C. "Phase II profile monitoring with binary data and random predictors", *Journal of Quality Technology*, 43, pp. 196–208 (2011).
[15] Samuel, T.R., Pignatiello, J.J. and Calvin, J.A. "Identifying the time of a step change with control charts", *Quality Engineering*, 10, pp. 521–527 (1998).
[16] Samuel, T.R., Pignatiello, J.J. and Calvin, J.A. "Identifying the time of a step change in a normal process variance", *Quality Engineering*, 10, pp. 529–538 (1998).
[17] Pignatiello, J.J. and Samuel, T.R. "Identifying the time of a step change in a Poisson rate parameter", *Quality Engineering*, 10, pp. 673–681 (1998).
[18] Pignatiello, J.J. and Samuel, T.R. "Estimation of the change point of a normal process mean in SPC applications", *Quality Technology*, 33, pp. 82–95 (2001).
[19] Perry, M.B. and Pignatiello, J.J. "Estimation of the change point of a Poisson rate parameter with a linear trend disturbance", *Quality and Reliability Engineering International*, 22, pp. 371–384 (2005).
[20] Perry, M.B. and Pignatiello, J.J. "Estimation of the change point of a normal process mean with a linear trend disturbance", *Quality Technology and Quantitative Management*, 3, pp. 101–115 (2006).
[21] Amiri, A. and Khosravi, R. "Estimating the change point of cumulative count of conforming control chart under a drift", *Scientia Iranica*, 19, pp. 856–861 (2012).
[22] Sharafi, A., Aminnayeri, M. and Amiri, A. "Identifying the time of step change in binary profiles", *The International Journal of Advanced Manufacturing Technology*, 63, pp. 209–214 (2013).
[23] Amiri, A. and Allahyari, S. "Change point estimation methods for control charts post-signal diagnostics: a literature review", *Quality and Reliability Engineering International*, 28, pp. 637–685 (2012).
[24] Albert, A. and Anderson, J.A. "On the existence of maximum likelihood estimates in logistic regression models", *Biometrika*, 71, pp. 1–10 (1984).
[25] McCullagh, P. and Nelder, J.A., *Generalized Linear Models*, 1st Edn., Chapman & Hall, London, UK (1989).

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