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USING THE ORTHONORMALIZED B-SPLINES TO SOLVE THE FREDHOLM INTEGRAL EQUATIONS

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Abstract. In this paper, we orthonormalize the B-spline functions by Gram-schmidt algorithm and then use them in solving the linear Fredholm integral equations. We prove the convergence of this method and obtain order of the error. And finally some examples are given to show accuracy of the method.

1. Introduction

In this paper, we consider the integral equations of the form

$$y(x) - \int_a^b K(x,t)y(t)dt = f(x), \qquad x\epsilon[a,b] \tag{1.1}$$

where $f \in C[a,b]$ and $K \in C([a,b] \times [a,b])$.
Here we use the B-spline functions [1] in the orthonormalized form [2] to convert the integral equation to the remarkably simple system of algebraic equations and obtain an approximate solution of the general form of linear Fredholm integral equations. We also prove convergence of the method.

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2. Solving Fredholm Integral Equation(FIE)

Consider the approximate form of the equation (1.1)as

$$y_n(x) - \int_a^b K_n(x,t)y_n(t)dt \simeq f_n(x), \qquad x\epsilon[a,b]. \tag{2.1}$$

We approximate $K(x,t)$ by the degenerate form

$$K_n(x,t) \simeq \sum_{i=1}^{n+k}\sum_{j=1}^{n+k} k_{ij}u_i(x)u_j(t) \tag{2.2}$$

where k is the order of B-splines and $u_r : r = 1,2,...,n+k$ are the orthonormalized form of $B_i^k : i = -k, -k+1, ..., n-1$, and $k_{ij} = ((k(x,t), u_i(x)), u_j(t)), \quad i,j = 1,2,...,n+k,$
by using orthonormal property of $\{u_r\}$. We also use the approximate form

$$f_n(x) \simeq \sum_{i=1}^{n+k} f_i u_i(x) \tag{2.3}$$

for $f(x)$ with $f_i = (f,u), \quad i = 1,2,...,n+k.$
Then by substituting from (2.2) and (2.3) in (2.1) we obtain

$$y_n(x) - \sum_{i=1}^{n+k}\sum_{j=1}^{n+k} k_{ij}u_i(x)\int_a^b u_j(t)y_n(t)dt = \sum_{i=1}^{n+k} f_i u_i(x) \tag{2.4}$$

or

$$y_n(x) = \sum_{i=1}^{n+k}\sum_{j=1}^{n+k} k_{ij}\gamma_j u_i(x) + \sum_{i=1}^{n+k} f_i u_i(x) \tag{2.5}$$

where

$$\gamma_r = \int_a^b u_r(t)y_n(t)dt, \qquad r = 1,2,...,n+k. \tag{2.6}$$

Now by substituting from (2.5) in (2.6) and setting

$$\omega_{ri} = \int_a^b u_r(t)u_i(t)dt, \qquad r,i = 1,2,...,n+k \tag{2.7}$$

we get the linear system

$$\gamma_r - \sum_{i=1}^{n+k}\sum_{j=1}^{n+k} k_{ij}\omega_{ri}\gamma_j = \sum_{i=1}^{n+k} f_i\omega_{ri} \qquad r = 1,2,...,n+k \tag{2.8}$$

for the unknowns $\gamma_1, ..., \gamma_{n+k}$.
The orthonormal property of $u_1, u_2, ..., u_{n+k}$ implies that $\omega_{ri} = \delta_{ri}$,

therefore the system (2.8) takes the simple form

$$(1-k_{rr})\gamma_r - \sum_{j=1,j\neq r}^{n+k} k_{rj}\gamma_j = f_r, \qquad r=1,2,...,n+k. \tag{2.9}$$

By solving this system the approximate solution of the integral equation (1.1) is given by

$$y_n(x) = \sum_{i=1}^{n+k}\sum_{j=1}^{n+k} k_{ij}\gamma_j u_i(x) + f(x). \tag{2.10}$$

3. Error bound and convergence

In this section, we obtain an error bound for the presented method. To this end we prove the following results [1].

Lemma 3.1. *Let the function $f \in C^{k+1}[a,b]$ and $\{u_1, u_2, ..., u_{n+k}\}$ be the orthonormalized system of $\{B^k_{-k}, ..., B^k_{n-1}\}$ and let $f_n = \sum_{i=1}^{n+k} f_i u_i$ be approximation of f with $f_i = (f, u_i)$, then there exist a constant C such that*

$$\|f - f_n\| \leq Ch^{k+1}\|f^{(k+1)}\|. \tag{3.1}$$

Lemma 3.2. *Let $K(x,t) \in C^{k+1}([a,b]\times[a,b])$ and $K_n(x,t)$ be approximation of K defined by (2.2), then there exists a constant C such that*

$$|K(x,t) - K_n(x,t)| \leq Ch^{k+1}. \tag{3.2}$$

Lemma 3.3. *The sequence $A_n : C[a,b] \longrightarrow C[a,b]$ defined by*

$$A_n y(x) = \int_a^b K_n(x,t)y(t)dt \tag{3.3}$$

with given $K_n(x,t)$ by (2.2) is a collectively compact sequence and pointwise convergent $A_n y \longrightarrow Ay, n \longrightarrow \infty$ for all $y \in C[a,b]$.

Lemma 3.4. *Under the assumptions of theorem 10.9, p. 168 of [4], we have the error estimate*

$$\|y_n - y\| \leq C\{\|(A_n - A)y\| + \|f_n - f\|\} \tag{3.4}$$

for all sufficiently large n and some constant C.

Theorem 3.5. *Assume that the kernel K and the right hand side f in equation (1.1), are $k+1$ times continuously differentiable, then y_n approximate the solution y of (1.1) with order $O(h^{k+1})$, i.e.*

$$\|y_n - y\| = O(h^{k+1}) \tag{3.5}$$

4. Numerical examples

The following example is given to clarify accuracy of the presented method.

Example 1([4], p. 190, p. 204).

$$y(x) - \frac{1}{2}\int_0^1 (x+1)e^{-xt}y(t)dt = e^{-x} - \frac{1}{2} + \frac{1}{2}e^{-(x+1)} \quad , \quad 0 \le x \le 1.$$

The exact solution is $y(x) = e^{-x}$.
Table 1 shows the absolute errors between the approximate and the exact solution at the points $x = 0,\ 0.25,\ 0.5,\ 0.75,\ 1$ and for various values of stepsize h.

Table 1 : Numerical results of Example 1.

h	k	$x=0$	$x=0.25$	$x=0.5$	$x=0.75$	$x=1$
0.2500	3	$0.41e-5$	$0.44e-5$	$0.61e-5$	$0.90e-5$	$0.13e-4$
0.2000	3	$0.17e-5$	$0.21e-5$	$0.29e-5$	$0.42e-5$	$0.59e-5$
0.1250	3	$0.36e-6$	$0.27e-6$	$0.27e-6$	$0.59e-6$	$0.59e-6$
0.1000	2	$0.14e-6$	$0.98e-7$	$0.23e-6$	$0.21e-6$	$0.34e-6$
0.0625	2	$0.18e-7$	$0.20e-7$	$0.14e-7$	$0.90e-7$	$0.10e-5$
0.0500	2	$0.14e-8$	$0.10e-8$	$0.62e-7$	$0.21e-7$	$0.73e-6$

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