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An Optimality Probability Index for Correlated Multiple Responses

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In the real problems, there are many cases which have correlated quality characteristics so multiple response optimization can be more realistic if we can consider correlation structure of responses. In this study we propose a new method which uses multivariate normal probability to find the optimal treatment in an experimental design. Moreover, a heuristic method is used to find better factors' level in all possible combinations in the designs with large number of controllable factors and their levels. Some simulated numerical examples and a real case were studied by the proposed approach and the comparison of the results with previous methods show efficiency of the proposed method.

Keywords Multiple response optimization; Correlation; Design of experiments; Multivariate normal probability.

Mathematics Subject Classification 62K99; 65C60.

1. Introduction

Multiple response optimization can be divided into two general categories: independent responses systems and correlated responses systems. In independent responses systems, response variables are entirely independent from each other and there are no correlations between them. For example, consider a process where its product is a piston. If the response variables are piston surface smoothness and piston height, it can be concluded that we are dealing with an independent multi-response system because, in fact, surface smoothness has no correlation with piston height. In these multi-response systems, we can consider each response independently and study changes in levels of control factors on each response without considering other response variables. According to this method, it is obvious that analysis and optimization (finding the best factor-level combination) stages are more simple in these systems rather than in correlated responses systems. To assure the efficiency of the model in analyzing independent multi-response system design, it is necessary to

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check the normality assumption of the responses. In a process with correlated responses, it is not possible to analyze and study each response separately because it may result in wrong interpretations. Consider the piston production example which response variables are weight and piston diameter respectively. It is obvious that we cannot consider them as two completely independent responses. Note that in analyzing these problems under classical experimental design methods, the realized values for response variables should necessarily have multivariate normal distribution in order to satisfy the adequacy of model. Due to the difficulty of analysis and optimization of experiments in these systems, in most articles correlation between response variables is usually ignored (for example, Ramakrishnan and Karunamoorthy, 2006, Tong eta al., 2007; Bashiri and Hejazi, 2009). For this reason, optimization approaches for correlated multi-response systems have lack of variety.

Approaches for optimization of independent multi-response problems can be divided into several general categories. One of these categories is dealing with complicated mathematical statistical models. Such approaches include the following: Khuri and Conlon (1981), Logothetis and Haigh (1988), and Pignatiello and Joseph (1993) who used polynomial regression model for multi-response optimization. Tong and Su (1997) developed a method based on applying fuzzy set theory for optimization of multi-response production process. Ames et al. (1997) used response surface methodology for solving multi-response problems. Another category consists of heuristic and metaheuristic algorithms and neural network based methods for multi-response optimization. For example, Jeyapaul et al. (2005) provided an integrated approach using signal to noise ratio and genetic algorithm for optimization of multi-response problems. Su and Hsieh (1998) and Tong and Hsieh (2000) used artificial neural network. These methods cannot be applied equivalently in every multi-response problem. In another category researcher first converts all responses to one process performance index (PPI) and then optimizes the process considering this index. Following examples are using such approaches: Derringer and Suich (1980) modified desirability function to optimize several response variables simultaneously. Shiau (1990) and Tai et al. (1992) considered weighted signal to noise ratio as process performance index. Pan et al. (2007) and Haq et al. (2007) used fuzzy dependency analysis method for obtaining PPI. Ramakrishnan and Karunamoorthy (2006) suggested multiple response signal to noise ratio. Tong et al. (2007) used VIKOR method, which is adaptive ranking method in multi-criteria decision making (MCDM), for multi-response optimization. Bashiri and Hejazi (2009) studied multi-response optimization from the MADM point of view using various methods such as: TOPSIS, which operate based on the shortest distance from ideal solution and the largest distance from the negative ideal solution, ELECTRE III, VIKOR, PROMETHEE II, which uses preference functions for indicating the difference between cases. Gauri and Pal (2010) implemented five methods—WSN, GRG, MRSN, VIKOR, and WPC— on three data sets. They mentioned methods that assume responses are uncorrelated. This article is based on the last mentioned category but considering the correlation structure as well.

Optimization of correlated multiple response problems has been discussed less than independent ones. Chiao and Hamada (2001) found the best probability of being responses in a specification region. Maghsoodloo and Chang (2001) developed the quadratic loss function and signal to noise ratios for a bivariate response when both quality characteristics were from the same type. Then, Maghsoodloo and Huang (2001) studied on mixed bivariate responses and developed quadratic loss function and signal to noise ratios for them. Ozdemir and Maghsoodloo (2004) extends quadratic quality loss function and signal to noise ratios for trivariate cases. Ko et al. (2005) proposed a new loss function method which accommodates robustness, quality of predictions and bias in a single framework.

Table 1
A classification of some previous studies for optimization of correlated multiple response problems

Method	Article(s)	Strength(s)	Weakness(es)
Probability of being in a specification region	Chiao and Hamada (2001)	Stochastic results, easy to understand basis	Results depend on specification region, Do not consider distance to target, Difficult to use when number of responses increases
Quality Loss function	Maghsoodloo and Chang (2001); Maghsoodloo and Huang (2001); Ozdemir and Maghsoodloo (2004), Ko et al. (2005)	Easy to use, consider robustness	Deterministic results, Difficult to understand basis
Principal component analysis	Antony (2000), Liao (2006), Datta et al. (2009)	Easy to use	Deterministic results, Some information loses, Optimization direction is changed and indefinite after transformation
Double-exponential desirability function	Wu (2005)	Uses both quality loss and desirability functions concept	Deterministic results, Results depend on specification region, Difficult to understand basis
Optimality probability index (OPI)	This paper	Stochastic results, Easy to understand basis, Considers distance to target,	difficult to use when number of responses increases

Some articles applied principal component analysis (PCA) to transform some correlated responses to the same number or fewer independent responses. For example, Antony (2000) used the first PC to solve the problem but Liao (2006) studied the problem by considering all PC's and proposed weighted principal component method. Datta et al. (2009) utilized genetic algorithm after performing PCA method. Wu (2005) proposed an approach based on the double-exponential desirability function which has been modified by considering Taguchi's loss function, to optimize the correlated multiple quality characteristics. Table 1 shows classification of some previous studies for optimization of correlated multiple response problems considering their strengths and weaknesses.

As shown in Table 1, stochastic results considered as strength for Chiao and Hamada (2001)'s method, because responses in real problems are stochastic and so it is better to answer to these problems stochastically it means that a treatment is an optimum one with a probability. In the other methods lack of this advantage can be considered as weakness. Nevertheless, Chiao and Hamada (2001)'s method needs to specify specification limits for each response and moreover it cannot consider distance of responses from their targets. In this study, we proposed an approach based on multivariate normal probability to find the best factor-level combination of an experimental design with correlated responses without need to specify limits where distance to target are considered in calculations and each combination's optimality probability is computed by considering all other treatments. In the next section, we define model and estimate its parameters. Then we propose a heuristic algorithm to search and find the best treatment for the large-sized problems. For better comprehension, some numerical examples are presented in Sec. 3 and finally conclusions are presented in the last section.

2. Proposed Method

2.1. Model Statement

Suppose that we have an experimental design with n treatments and m normally distributed correlated responses. Our goal is to find the best factor-level combination to achieve optimum values for responses. For this purpose, we define a multivariate probability for each combination which shows the probability of being optimum in all responses between all the treatments. Equation (1) shows the optimality probability index (OPI) for treatment k:

$$OPI_k = \begin{cases} \prod_{\substack{i=1\\ i\neq k}}^{n} P(y_{1k} > y_{1i}, \ldots, y_{mk} > y_{mi}) \text{ for LTB Responses} \\ \prod_{\substack{i=1\\ i\neq k}}^{n} P(y_{1k} < y_{1i}, \ldots, y_{mk} < y_{mi}) \text{ for STB Responses} \\ \prod_{\substack{i=1\\ i\neq k}}^{n} P(|y_{1k} - t_1| < |y_{1i} - t_1|, \ldots, |y_{mk} - t_m| < |y_{mi} - t_m|) \text{ for NTB Responses} \end{cases}$$
$$k = 1, 2, \ldots, n \tag{1}$$

where y_{mi} is mth response corresponding to the ith treatment and the t_m is target value for mth response when its type is nominal the best (NTB).

In the case of larger the better (LTB) and smaller the better (STB) responses, when all responses have normal distribution, OPI_k is product of m multivariate normal probabilities. However, in the case of NTB responses, the best factor-level combination should have minimum absolute values of responses from their targets for all responses. So, the multivariate probability function should not necessarily have normal distribution. To overcome this problem, first we can transform NTB responses to STB type by using absolute values of target corrected responses. As mentioned before this transformation can violate the normality assumption. In such cases we should use another transformation on new STB responses to change their distribution to normal again. There are many problems with one or more characteristics which have different type from other ones. For example, suppose that we have a problem, with two correlated responses which have STB and LTB type, respectively. In such problems, we can use the $y = 1/x$ transformation where x is an LTB quality characteristic and obtained y is an STB one or use a mixed multivariate probability which can be obtained from Eq. (1).

2.2. Parameter Estimation

For calculating Eq. (1), we should estimate parameters for multivariate normal distribution, in each treatment. Equation (2), shows multivariate normal distribution where $Y = [y_1, y_2, \ldots, y_m]$, $\mathrm{M} = [\mu_1, \mu_2, \ldots, \mu_m]$ and $\Sigma = [\sigma_{y_i y_j}]$, $i = 1, 2, \ldots, m; j = 1, 2, \ldots, m$ are response vector, mean vector, and covariance matrix, respectively.

$$f(y_1, y_2, \ldots, y_m) = \frac{1}{(2\pi)^{\frac{m}{2}} |\Sigma|^{\frac{1}{2}}} e^{\left(-\frac{1}{2}(Y-M)'\Sigma^{-1}(Y-M)\right)} \tag{2}$$

Note that $|\Sigma|$ is determinant of covariance matrix and $(Y - M)'$ is transpose of $(Y - M)$.

The least square estimators for multivariate normal probability parameters are as follows:

$$\hat{\mu}_i = x\alpha_i,\ i = 1, \ldots, m \tag{3}$$

$$\log\left(\hat{\sigma}_i^2\right) = x\beta_i,\ i = 1, \ldots, m \tag{4}$$

$$\tanh^{-1}(\hat{\rho}_{ij}) = x\gamma_{ij},\ 1 \le i < j \le m \tag{5}$$

The logarithmic model for variances ensures positive values for them. Correlations are between -1 and 1 so we use the inverse hyperbolic tangent transformation (Rao, 1973) which is defined as:

$$\tanh^{-1}(\rho) = \frac{1}{2} \log \frac{(1+\rho)}{(1-\rho)} \tag{6}$$

By estimating the above parameters for responses corresponding to each treatment we can calculate optimality probability for each treatment of experimented or non-experimented ones with respect to other possible treatments by using Eq. (1).

2.3. Proposed Heuristic Algorithm

In the case of a problem with large number of controllable factors and their levels, calculating Eq. (1) for all factor-level combinations may be very time consuming. For better comprehension suppose that we want to calculate OPI for some problems with different number of controllable factors. Figure 1 shows estimation of computational time to solve such problems on notebook with an AMD E-350 processor and 4 gigabytes of RAM when all factors have 3 levels.

It is obvious that by increasing the number of factors, calculation time increases exponentially. So, a heuristic algorithm can be useful to overcome the computation time problem. The proposed approach tries to find the most probable treatment to be optimum in a first probability calculation and in each iteration finds it again. So solution space in each iteration decreases more and more and this strategy can reduce number of calculations. The proposed approach is described in Table 2 as a pseudo-code.

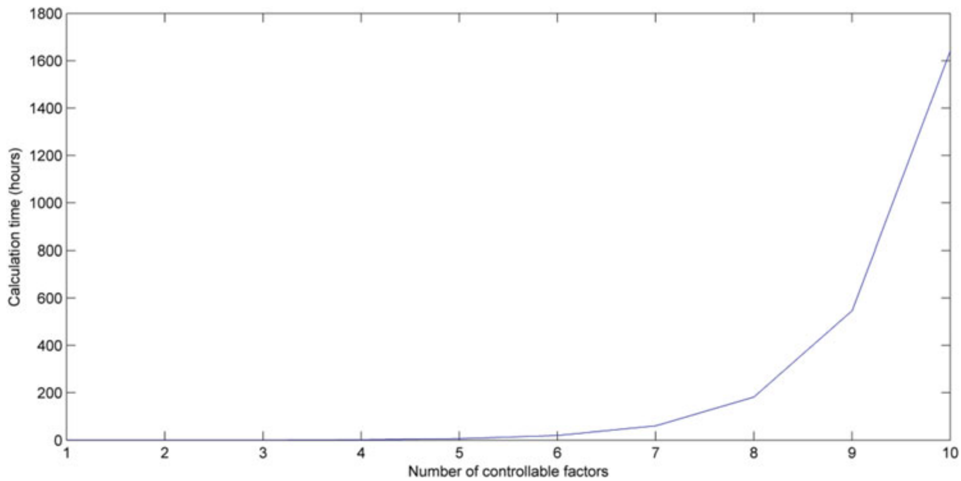

Figure 1. Computational time against number of controllable factors.

3. Numerical Examples

In this section, two numerical examples from previous articles were studied according to the proposed method and results were compared with results of referenced articles. Then a simulated numerical example with large number of factors is presented to show efficiency of the proposed heuristic algorithm.

Table 2
Pseudo-code of the proposed heuristic algorithm to find the best combination

Transform all responses to STB type;
  Find the estimation equations for the parameters of multivariate normal probability, $\hat{\mu}_i, \hat{\sigma}_i, \hat{\rho}_{ij}, i = 1,2,\ldots,m, j = 1,2,\ldots,m$ using Eqs. (3)–(5);
  select a random factor-level combination k;
  calculate multivariate normal probability parameters for responses in combination k;
  calculate $p_{k,j}(y_{1k} < y_{1j}, \ldots, y_{mk} < y_{mj})$ for $j = 1,2,\ldots,n$ where $j \neq k$;
  calculate Eq. (1) for combination k, p_k;
  set $k* = k$, and $p_k* = p_k$;
  set the number of desired combinations to check, l;
  select l combinations with minimum $p_{k,j}$ values and store in J;
  repeat
    for $k = J(1)$ to $J(l)$
    calculate multivariate normal probability parameters for responses in combination k;
    calculate $p_{k,j}(y_{1k} < y_{1j}, \ldots, y_{mk} < y_{mj})$ for $j = 1,2,\ldots,n$ where $j \neq k$;
    calculate equation (1) for combination k, p_k;
    if $p_k > p_k*$, set $k* = k$ and $p_k* = p_k$;
    next k;
    select l combinations with minimum $p_{k*,j}$ value and store in J;
    until p_k* do not changes.

Table 3
Values for means and variances of each response in each treatment on Example1

Treatment	A	B	C	D	μ_1	μ_2	μ_3	$\sigma^2{}_1$	$\sigma^2{}_2$	$\sigma^2{}_3$
1	1	1	1	1	52.8398	43.3660	64.1734	20.9679	7.0849	35.9961
2	2	1	2	1	20.4600	10.9455	32.8927	15.7183	12.8105	8.3774
3	1	2	2	1	37.9198	28.9401	53.6243	46.9146	29.8066	105.8522
4	2	2	1	1	34.5153	25.5751	48.0280	15.8422	7.8996	3.6614
5	2	2	2	2	47.7471	38.6355	61.5471	3.9807	2.5741	9.3844
6	1	2	1	2	48.9013	39.7489	61.5565	13.5287	15.1502	15.4528
7	2	1	1	2	31.8805	21.7594	42.3877	8.0884	7.5814	5.5111
8	1	1	2	2	36.4230	25.8339	50.4595	22.5505	10.2026	49.4551

3.1. *Example 1*

Consider an experimental design with three STB-type responses and four controllable factors extracted from Ozdemir and Maghsoodloo (2004). The design of experiment is 2^{4-1} fractional factorial design with resolution IV. Each response has four replicates in each treatment. Figure 2 illustrates normal probability plots with 95% confidence intervals for three STB responses values and it can be seen from p-values that we can't reject the hypothesis that responses values have normal distribution. Table 3 shows the means and variances values for each response and Table 4 shows correlations values

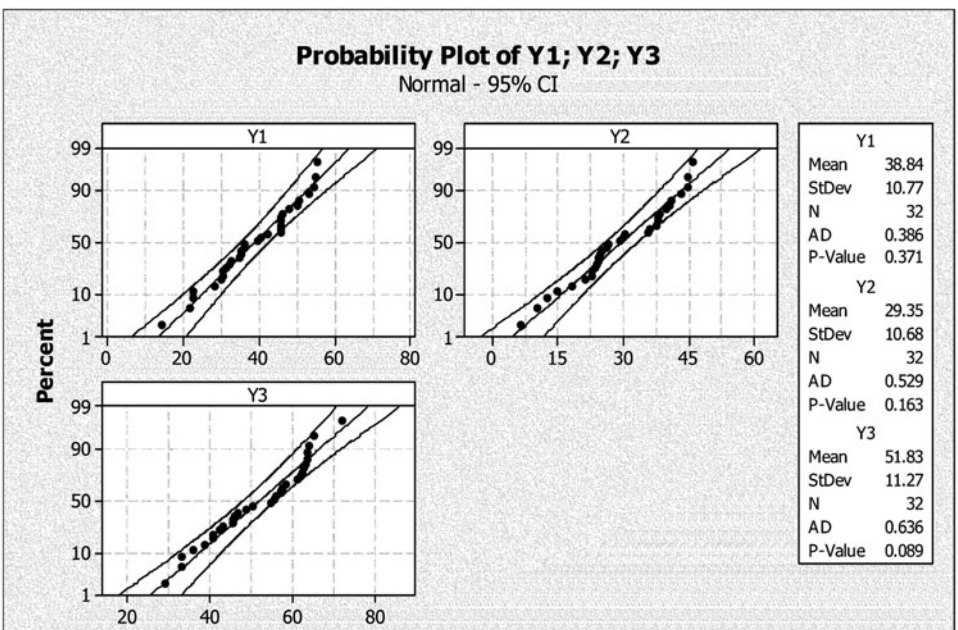

Figure 2. Normal probability plots with 95% confidence interval for Example 1 responses.

Table 4
Values for correlations between responses in each treatment on Example 1

Treatment	A	B	C	D	$\rho_{(1,2)}$	$\rho_{(1,3)}$	$\rho_{(2,3)}$
1	1	1	1	1	0.8873	−0.8793	−0.9796
2	2	1	2	1	0.8920	0.0439	0.2580
3	1	2	2	1	0.9630	0.3791	0.5988
4	2	2	1	1	0.9199	0.9708	0.9500
5	2	2	2	2	0.8168	−0.4368	0.1621
6	1	2	1	2	0.9749	0.3053	0.4820
7	2	1	1	2	0.8343	−0.2329	0.2833
8	1	1	2	2	0.9351	−0.8267	−0.9431

between two responses at each treatment. These values are obtained from equations (3)–(5).

In this example we want to select one of the implemented treatments as optimum so there is no need to estimate other treatments parameters. By calculating probability (6) for each of eight treatments, the optimum treatment can be found:

$$P_k = \prod_{\substack{i=1 \\ i \neq k}}^{8} P(y_{1k} < y_{1i}, y_{2k} < y_{2i}, y_{3k} < y_{3i}). \tag{7}$$

Table 5 shows the optimality probability indices and average quality losses values for each treatment. It can be seen that the factor-level combination (2,1,2,1) in treatment 2 is optimum by considering other treatments *OPIs*. The quality losses values from Ozdemir and Maghsoodloo (2004) confirms our result about first and second best treatments. However, it can be seen that combination (1,2,2,1) in treatment 3 which is the 3rd best combination based on *OPI*'s values, has 5th minimum quality loss average. Note that, the *OPIs* show the probability of being optimum for each treatment. So, when the treatment (2,1,2,1) which

Table 5
Optimality probability for each treatment in Example 1

Treatment	A	B	C	D	OPI_k	OPI based rank	Average quality losses (AQL)	AQL based rank
1	1	1	1	1	8.08×10^{-170}	8	52.9142	8
2	2	1	2	1	0.2187	1	8.1328	1
3	1	2	2	1	1.08×10^{-7}	3	29.353	5
4	2	2	1	1	7.13×10^{-10}	4	22.0571	3
5	2	2	2	2	9.98×10^{-64}	7	44.7216	6
6	1	2	1	2	2.93×10^{-34}	6	46.3093	7
7	2	1	1	2	1.97×10^{-6}	2	15.7000	2
8	1	1	2	2	1.04×10^{-16}	5	23.0897	4

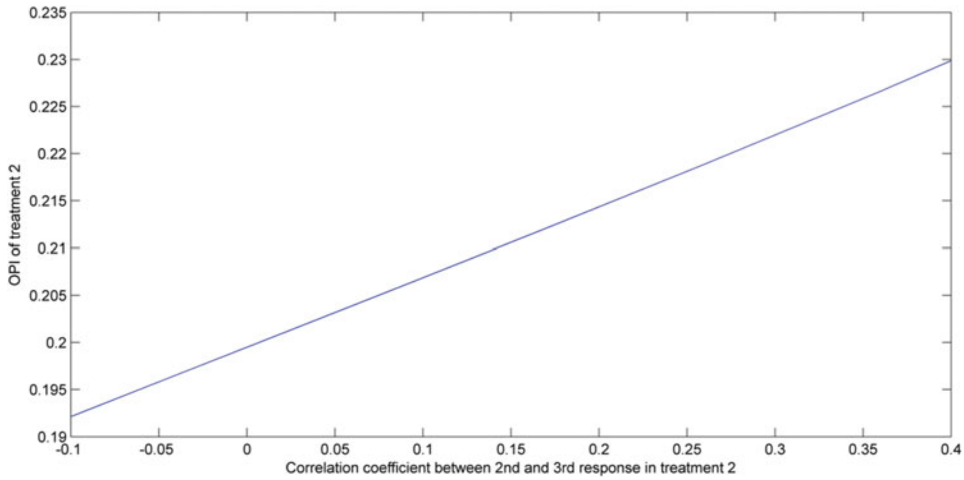

Figure 3. OPI changes for best treatment against correlation coefficient between response 2 and 3 in treatment 2.

has smaller quality losses value is optimum by probability of 0.2187, it is logical that other treatments can be optimum by probability of almost zero.

For better comprehension, a sensitivity analysis can be done. Figure 3 shows increase of OPI for best treatment when correlation coefficient between response 2 and 3 changes from −0.1 to 0.4 in treatment 2. To analyze effect of variance, suppose that variance of best treatment changes from 1 to 25. Figure 4 shows changes of OPI for best treatment when its variance shifts. It shows that optimality probability of a treatment will be decreased when its related responses variances is growing.

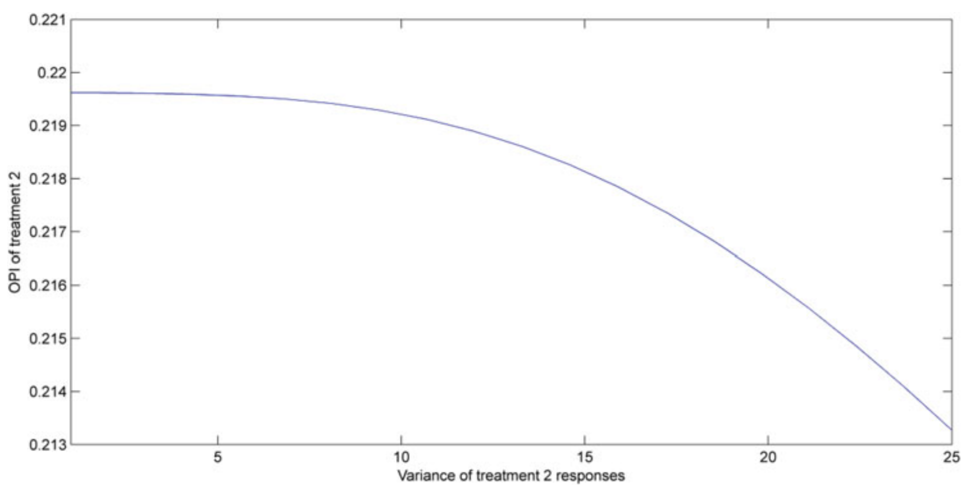

Figure 4. OPI changes for best treatment against variance of treatment 2.

Table 6
Optimality probability index and Chiao and Hamada (2001)'s probability for the three best treatments in Example 2

A	B	C	D	E	G	OPI_k	Chiao and Hamada (2001)'s probability
−1	−1	−1	−1	−1	1	0.2326×10^{-12}	0.5332
−1	1	−1	−1	−1	1	0.0058×10^{-12}	0.7216
−1	−1	1	−1	−1	1	0.0075×10^{-16}	0.5332

3.2. Example 2

As the second example, consider a design to minimize the imbalance of a plastic wheel cover component described by Chiao and Hamada (2001). There are two NTB quality characteristics: total weight (Y_1) and the balance of the component (Y_2) and seven controllable factors with two levels which are important to the component's balance. As mentioned before in Sec. 2, we should use Johnson transformation (Chou et al., 1988) to change NTB responses to STB types in order to use the proposed method. Equation 7 and 8 show these transformations for two responses where Z_1 and Z_2 are transformed STB type responses:

$$Z_1 = 0.387228 + 0.365089 \times \mathrm{Ln}\left(\frac{|Y_1 - \mu_1| - 0.0888751}{21.2924 - |Y_1 - \mu_1|}\right) \tag{8}$$

$$Z_2 = 0.122266 + 0.389592 \times \mathrm{Ln}\left(\frac{|Y_2 - \mu_2| - 0.0863844}{1.26185 - |Y_2 - \mu_2|}\right). \tag{9}$$

Then least square estimators for multivariate distribution parameters can be calculated using Eqs. (3)–(5) to write experimental factor models as follows:

$$\begin{aligned}
\hat{\mu}_1 &= 0.079 - 0.288x_4 + 0.643x_5 - 0.297x_7 \\
\hat{\mu}_2 &= -0.047 + 0.109x_3 + 0.241x_1 + 0.763x_5 - 0.486x_7 \\
\log\left(\hat{\sigma}_1^2\right) &= -1.060 - 0.390x_1 + 0.202x_4 - 0.267x_7 \\
\log\left(\hat{\sigma}_2^2\right) &= -0.870 - 0.282x_2 \\
\tanh^{-1}(\hat{\rho}_{1,2}) &= -0.106 + 0.559x_1 - 0.318x_5 - 0.494x_7
\end{aligned} \tag{10}$$

Table 6 shows three best factor-level combinations for this example. It is obvious from Eq. (9) that factor F is not significant and does not affect on probability value. The optimal combination by the proposed method is (-1,-1,-1,-1,-1,_,1) with optimality probability index of 0.2326×10^{-12} where an insignificant factor denoted by _. It can be seen that combination (-1,1,-1,-1,-1,_,1) which is the best combination in Chiao and Hamada (2001), has better index of Chiao and Hamada (2001)'s probability but it's optimality probability index is not large enough to be the optimal combination. Note that Chiao and Hamada (2001)'s probability shows the proportion of conformance to specification region and is very sensitive to defined upper and lower bounds of the responses. However, the *OPI* shows the probability of being the optimum treatment between other possible combinations.

Table 7
Calculated parameters in each treatment for Example 3

Treatment	A	B	C	D	E	F	G	H	I	μ_1	μ_2	σ^2_1	σ^2_2	$\rho_{(1,2)}$
1	1	1	1	1	1	1	1	1	1	4.1470	102.2243	0.0108	15.1017	−0.2968
2	1	1	1	1	2	2	2	2	2	3.3581	112.8818	0.0068	1.9950	−0.5470
3	1	1	1	1	3	3	3	3	3	3.8452	114.4757	0.0042	6.3085	0.8369
4	1	2	2	2	1	1	1	2	2	2.6830	103.8821	0.0079	8.8593	−0.2996
5	1	2	2	2	2	2	2	3	3	2.3656	95.9612	0.0020	1.8893	−0.7966
6	1	2	2	2	3	3	3	1	1	2.7403	109.2219	0.0028	3.4271	−0.8795
7	1	3	3	3	1	1	1	3	3	4.2255	102.5917	0.0204	15.2438	0.9620
8	1	3	3	3	2	2	2	1	1	3.2540	106.0403	0.0034	7.8308	0.3466
9	1	3	3	3	3	3	3	2	2	2.1211	100.5824	0.0054	2.8403	0.4179
10	2	1	2	3	1	2	3	1	2	3.1943	103.1929	0.0040	1.3277	−0.2117
11	2	1	2	3	2	3	1	2	3	4.3310	109.2780	0.0192	2.1912	0.3644
12	2	1	2	3	3	1	2	3	1	3.1498	106.6018	0.0019	7.9685	0.6525
13	2	2	3	1	1	2	3	2	3	3.9268	112.3448	0.0011	1.3944	−0.3144
14	2	2	3	1	2	3	1	3	1	3.1411	108.5867	0.0073	9.6146	−0.0389
15	2	2	3	1	3	1	2	1	2	3.5400	98.00457	0.0045	3.2939	−0.0311
16	2	3	1	2	1	2	3	3	1	3.6162	107.0205	0.0036	0.6142	−0.9760
17	2	3	1	2	2	3	1	1	2	3.3618	97.19684	0.0013	12.3763	0.8491
18	2	3	1	2	3	1	2	2	3	3.1058	106.5303	0.0060	6.0686	−0.9593
19	3	1	3	2	1	3	2	1	3	4.5658	102.2799	0.0122	1.0111	0.5382
20	3	1	3	2	2	1	3	2	1	3.5561	101.0539	0.0040	6.8441	0.1881
21	3	1	3	2	3	2	1	3	2	4.8375	101.3454	0.0075	3.4844	−0.0213
22	3	2	1	3	1	3	2	2	1	3.6075	103.2989	0.0051	0.0620	0.9199
23	3	2	1	3	2	1	3	3	2	3.7011	96.95306	0.0043	3.2635	−0.9215
24	3	2	1	3	3	2	1	1	3	3.9105	104.8708	0.0003	5.3679	0.0167
25	3	3	2	1	1	3	2	3	2	3.2120	107.8587	0.0044	17.8818	0.9324
26	3	3	2	1	2	1	3	1	3	2.6008	103.7729	0.0004	0.7953	0.5587
27	3	3	2	1	3	2	1	2	1	4.6149	102.5406	0.0093	1.3074	−0.2479

Table 8
Optimality probability for the three best treatments in Example 3

Rank	A	B	C	D	F	G	H	I	OPI_k
1	1	3	2	1	3	3	1	1	4.8997×10^{-128}
2	1	3	2	1	3	2	1	1	1.0896×10^{-144}
3	1	3	2	1	3	1	1	1	3.1330×10^{-169}

3.3. Example 3

In previous examples because of problems size the probability calculations need a negligible computational time and it was not necessary to use a heuristic search approach. Now we consider a simulated design which has nine controllable factors with three levels for each of them and two STB correlated responses. Table 7 shows calculated parameters for each treatment.

The least square estimators for multivariate normal distribution parameters can be found as below:

$$\begin{aligned}
&\hat{\mu}_1 = 6.518 + 0.494x_1 - 0.271x_2 - 1.758x_3 + 0.447x_3^2 + 0.643x_5 + 0.337x_8x_9 \\
&\hat{\mu}_2 = 114.104 + 2.241x_4 - 7.825x_6 - 7.822x_8 + 4.587x_6x_8 \\
&\log\left(\hat{\sigma}_1^2\right) = -2.011 - 0.054x_1x_2 + 0.091x_7x_9 + 0.065x_8x_9 \qquad (11) \\
&\log\left(\hat{\sigma}_2^2\right) = 1.701 - 0.216x_1 - 0.223x_7 - 0.077x_4x_6 \\
&\tanh^{-1}(\hat{\rho}_{1,2}) = -1.132 + 1.165x_1x_6 + 0.125x_2x_3
\end{aligned}$$

Solving such problem on a notebook with an AMD E-350 dual-core processor and 4 gigabytes of RAM can take lots of time about 550 h. So, it is better to use the proposed heuristic approach. The algorithm was coded in MATLAB and after about 576 s as computational time, it shows that optimum combination is (1,3,2,1,_,3,3,1,1) with optimality probability of 4.8997×10^{-128} between 6561 possible combinations. Table 8 shows optimality probability values for three best combinations.

4. Conclusions

Multiple response optimization problems are complicated when there is correlation between responses. In these problems, ignoring the correlation structure can undermine our method, so we should use some techniques which can consider the correlation between responses. In this study, we proposed a method for problems which can find the best treatment by calculating a multivariate normal probability. A heuristic algorithm was presented to find the best factor-level combinations in problems with large number of factors. The results showed the efficiency of the proposed method comparing to other existing approaches. The case of NTB responses without transforming to STB ones can be studied as a future research. Moreover, non-normal responses can be considered in the future studies.

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