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# OPTIMIZING THE REMESHING PROCEDURE BY COMPUTATIONAL COST ESTIMATION OF ADAPTIVE FEM TECHNIQUE

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**Abstract.** The objective of adaptive techniques is to obtain a mesh which is optimal in the sense that the computational costs involved are minimal under the constraint that the error in the finite element solution is acceptable within a certain limit. But adaptive FEM procedure imposes extra computational cost to the solution. If we repeat the adaptive process without any limit, it will reduce efficiency of remeshing procedure. Sometimes it is better to take an initial very fine mesh instead of multilevel mesh refinement. So it is needed to estimate the computational cost of adaptive finite element technique and compare it with the FEM computational cost. The remeshing procedure can be optimized by balancing these computational costs.

## 1 INTRODUCTION

Optimization is an important tool in decision science and in the analysis of physical systems. In an industrial context, the aim of the mechanical simulations in engineering design is not only to obtain greatest quality but more often a compromise between the desired quality and the computation cost (CPU time, storage, software, competence, human cost, computer used). The accuracy in numerical analysis of finite element solution strongly depends on the quality of FE mesh. Various mesh refinement techniques have been implemented to provide a physically acceptable solution. The objective of adaptive technique is to obtain a mesh which is optimal in the sense that the computational costs are minimal under the constraints, and the error of finite element solution is acceptable within a certain limit. The error estimation in numerical computations is obviously as old as the numerical computations themselves. The very first paper in error estimation was reported by Richardson [1] for practical computations utilizing finite differences. The process of error estimation in finite element analysis was originally introduced by Babuska and Rheinboldt [2]. This process considers local residuals of the numerical solution. By investigating the residuals occurring in a patch of elements, or even in a single element it becomes possible to estimate the errors which arise locally, usually in the norm of energy. In the error estimation process two main aims are followed; firstly, the determination of error in the mesh and secondly, the reduction of error to an acceptable value by adaptive mesh refinement. Various error estimators are proposed in the literature [2], which can be divided mainly into two categories. A priori error estimation is based on the knowledge of characteristics of the solution and provides qualitative information about the asymptotic rate of convergence as the number of degrees of freedom goes to infinity. A posteriori error estimation employs the solution obtained by the numerical analysis, in addition to a priori assumptions about the solution. This method can provide quantitatively accurate measures of the discretization error, while a priori estimate method cannot. In the present study, a posteriori error estimator developed by Zienkiewicz and Zhu [3] is employed by using an h-refinement adaptive procedure. Despite of simplicity of this error estimator, it is reasonably accurate. Bellenger and Coorevits [4] proposed the use of alternative mesh refinement criteria based on: prescribed number of elements with maximum accuracy, prescribed CPU time with maximum accuracy and prescribed memory size with maximum accuracy.

## 2 ERROR ESTIMATION AND ADAPTIVE MESH REFINEMENT

The criterion for determining this optimal mesh is the value of error in approximated finite element solution. According to the adaptive mesh refinement technique the optimal mesh is obtained by keeping this error within prescribed bounds. For most problems there is not exact solution, so exact error is not available. An improved solution is used instead of exact solution to compute an estimated error. The finite element analysis with displacements results in piecewise continuous stresses, while actual stresses may be continuous throughout the domain. By smoothing the piecewise continuous stresses, an improved solution can be obtained. In order to obtain an improved solution, the nodal smoothing procedure is performed using the weighted superconvergent patch recovery (WSPR) technique, proposed by Khoei et al. for cohesive zone model [5] and ductile crack growth [6]. Having computed the improved stresses, the error can be approximated by

$$\mathbf{e}_\sigma \approx \mathbf{e}_\sigma^* = \boldsymbol{\sigma}^* - \hat{\boldsymbol{\sigma}} \tag{1}$$

where $\mathbf{e}_\sigma$ is the exact error and $\mathbf{e}_\sigma^*$ the estimated error. Since the pointwise error becomes locally infinite in critical points, such as crack tip, the error estimator can be replaced by a global parameter using the *L2* norm of error defined as:

$$\|\mathbf{e}_\sigma\| = \|\boldsymbol{\sigma}^* - \hat{\boldsymbol{\sigma}}\| = \left(\int_\Omega (\boldsymbol{\sigma}^* - \hat{\boldsymbol{\sigma}})^{\mathrm{T}} (\boldsymbol{\sigma}^* - \hat{\boldsymbol{\sigma}})\, \mathrm{d}\Omega\right)^{\frac{1}{2}} \tag{2}$$

In adaptive mesh refinement, the *L2* norm for each element is a more desirable quantity to optimize the mesh. By changing the whole domain to each element domain, this quantity is achievable. Hence, the square of *L2* norm of the overall domain can be obtained by summing element contributions, i.e.

$$\|\mathbf{e}_\sigma\|^2 = \sum_{i=1}^{m} \|\mathbf{e}_\sigma\|_i^2 \tag{3}$$

where $i$ represents an element contribution and $m$ is the total number of elements. To normalize the value of error norm, it is divided to the state variable (such as stress) norm. Thus, the overall percentage error is defined as

$$\theta = \frac{\|\mathbf{e}_\sigma\|}{\|\hat{\boldsymbol{\sigma}}\|} \tag{4}$$

This relative error norm can be used in the mesh refinement procedure. The remeshing procedure chosen here is based on $h$-adaptive mesh refinement, in which the polynomial order of shape functions remains constant during successive mesh refinements. The distribution of error norm across the domain indicates which portions need refinement and which other parts need de-refinement, or coarsening elements. Since the total error permissible must be less than a certain value, it is a simple matter to search the design field for a new solution in which the total error satisfies this requirement. In fact, after remeshing each element must obtain the same error and the overall percentage error must be less than the target percentage error, i.e.

$$\theta \le \theta_{\text{aim}} = \frac{\|\mathbf{e}_\sigma\|_{\text{aim}}}{\|\hat{\boldsymbol{\sigma}}\|} \tag{5}$$

The size of elements in new mesh depends on the relative error and the rate of convergence. The rate of convergence of standard elements is proportional to the order of shape functions, however, in the case of crack tip problems, it is proportional to the order of singularity. Thus, if $h$ represents the size of element and λ denotes the rate of convergence, the new element size can be obtained as

$$(h_i)_{\text{new}} = \left[ \frac{\left(\|\mathbf{e}_\sigma\|_i\right)_{\text{aim}}}{\left(\|\mathbf{e}_\sigma\|_i\right)_{\text{old}}} \right]^{1/\lambda} (h_i)_{\text{old}} \tag{6}$$

After indicating the size of elements from Eq. (6), a mesh satisfying the requirements will be finally generated by an efficient mesh generator which allows the new mesh to be constructed according to a predetermined size. In order to prevent mesh generation difficulties due to very small and large elements, the element size is limited by an upper and a lower bound.

## 3 NUMERICAL OPTIMIZATION OF REMESHING

Although adaptive remeshing procedure reduces the discretization error of the solution but if we repeat the adaptive process without any limit, it will reduce efficiency of remeshing procedure. Thus we should stop remeshig process at a certain step. Firstly, we estimate the number of nodes in each remeshing step. For this purpose an extrapolation is carried out on available data from last remeshing steps. In this study, power extrapolation ($y = An^x$) is used for prediction of next steps. Now we estimate the reduction of error in next steps. It is obvious that the estimated error converges to a certain error in last steps of remeshing. Therefore more remeshing steps are not efficient. An extrapolation can be used again to see how error reduces with increasing the degrees of freedom of model. The most important part of numerical optimization of remeshing procedure is the control of computational cost. It consists of remeshing procedure and FEM computational cost. Flops (Floating-point Operations Per Second) is a useful parameter for estimating the computational cost of a procedure. Each operation takes some flops depending on the complexity of the operation. The computational cost grows drastically in the last steps of remeshing process. This growth can be predicted by an extrapolation function. It is evident that final steps of remeshing are not efficient, because of low error reduction and high computational cost. Thus we can define an inefficiency parameter for each remeshing step as the required flops for 1% error reduction in that remeshing step. If this inefficiency parameter exceeds a predefined limit we will stop the remeshing process.

## 4 NUMERICAL SIMULATION RESULTS

In order to demonstrate the performance of proposed optimization strategy an example is analyzed numerically. The finite element model has been implemented using the standard isoparametric linear triangular elements. In addition, various uniform and adaptive mesh refinements are implemented to investigate the efficiency of error estimation and mesh refinement procedures. The entire process of simulation has been automatically performed without user intervention. This example presents a rectangular plate with a circular hole subjected to a uniform load at one edge. The material properties are as follows; $E = 2.1\times10^6$ kg/cm$^2$, $\nu = 0.3$. Two different simulations have been performed here. The first simulation has been carried out using a uniform fine FE mesh with no refinement. The next simulations correspond to adaptive FE analyses. Fig. 1 presents the uniform and adapted meshes during the adaptive remeshing.

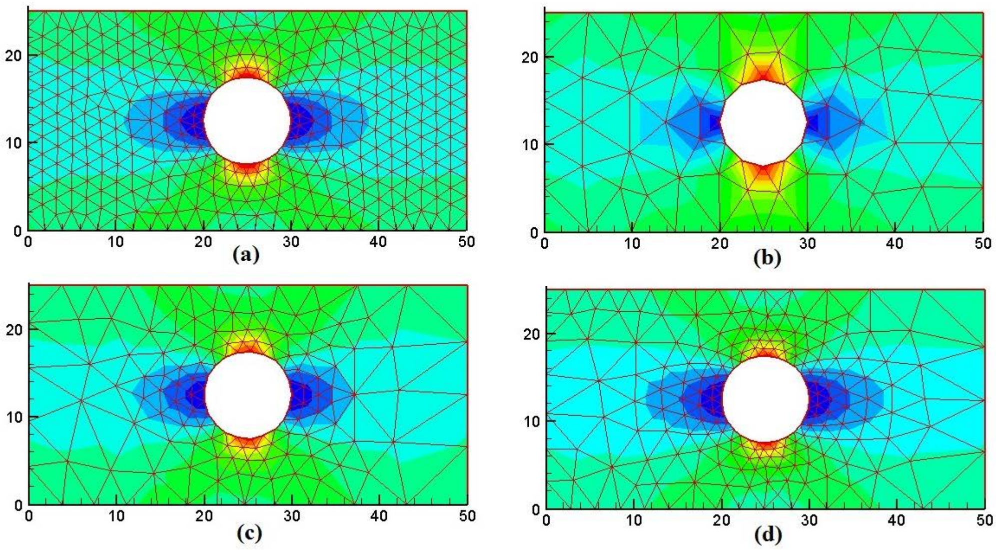


**Figure 1**: (a) The uniform and (b-d) adapted meshes during adaptive remeshing procedure

In the simulation with fine uniform mesh the estimated error was 9.6%, while the total flops was approximately $2.2\times10^6$. On the other hand, in adaptive remeshing procedure we have approached to 10% margin of error in the third step of remeshing. If we sum all flops in these three steps, it will result $2.2\times10^6$ flops. It shows that the adaptive remeshing procedure improves the mesh quality without any extra computational cost. If we want to continue the remeshing process in next steps, we should predict the number of nodes, error and flops in next steps. If we extrapolate these parameters from the data available for the three previous steps with a power function, we can predict them for the next steps. Fig. 2 shows the extrapolated function for predicting the number of nodes in different remeshing steps. Fig. 3 shows how the estimated error reduces as the number of nodes increase. It is obvious that we have less efficiency in error reduction for next remeshing steps. Finally, the growth rate of flops in different remeshing steps is shown in Fig. 4.

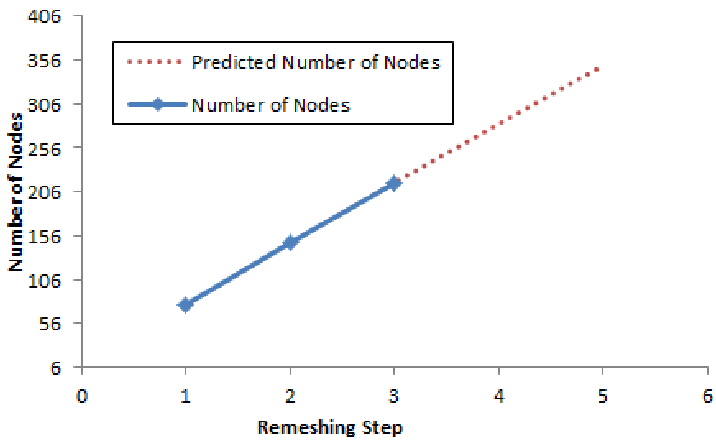


**Figure 2**: Predicted number of nodes in different remeshing steps

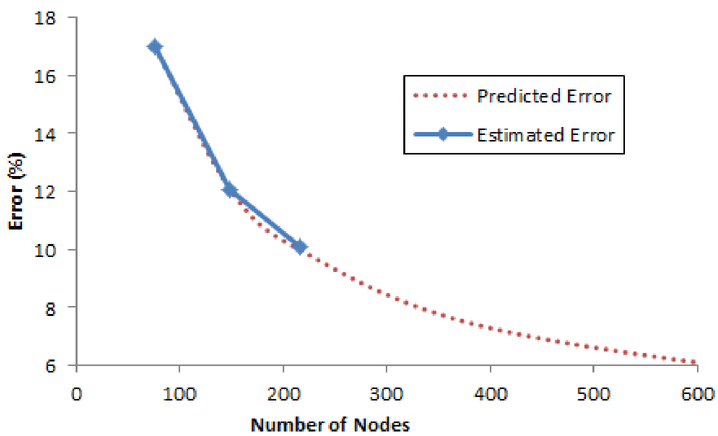


**Figure 3**: The variation of predicted error with number of nodes

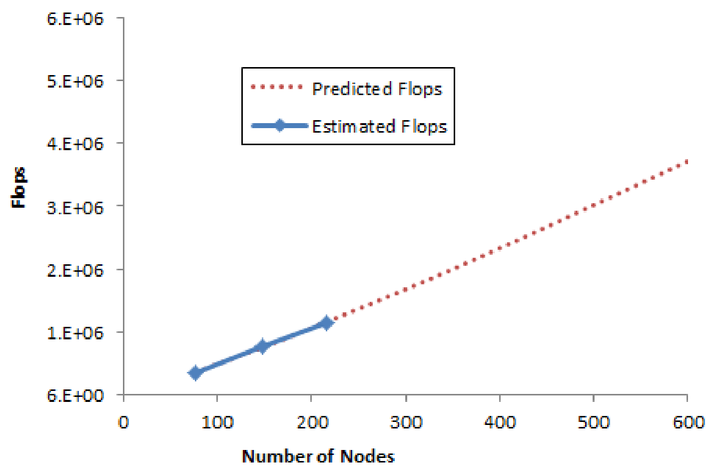


**Figure 4**: The variation of flops with number of nodes

It can be concluded from these charts that we achieve low error reduction with high computational cost in last remeshing steps. These results are summarized in Table 1. Thus, we should stop adaptive remeshing procedure in a certain step to optimize this process.

**Table 1**: Comparison of predicted error and flops in different remeshing steps

| Remeshing Step | Error | Flops | Flops for 1% error reduction |
|---|---|---|---|
| 1 | 16.92 | 360402 | - |
| 2 | 12.12 | 755861 | 82334 |
| 3 | 9.99 | 1160256 | 189974 |
| 4 | 8.70 | 1575979 | 322765 |
| 5 | 7.89 | 1959776 | 471370 |

In this example we accept 300,000 flops for 1% error reduction. Table 1. shows that in fourth remeshing step flops for 1% error reduction exceeds the specified limitation and we will stop the remeshing procedure in third step.

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