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ORIGINAL ARTICLE

Monotonic change point estimation in the mean vector of a multivariate normal process

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Abstract When a control chart sounds the alarm that the process is out of control (OC), the process will be paused and specialists will start the procedure of finding the root cause(s) that made the process out of control. Knowing the time of change will substantially aid the process engineer to figure out the assignable causes and solve the problem sooner, so the time, energy, and costs spent to implement corrective actions will be considerably reduced. Maximum likelihood estimator (MLE) as one of the statistical technique is frequently used for estimating the change point time. In this paper, an MLE is derived to estimate the time of first change in the mean vector of a multivariate normal process when the type of change is monotonic. The performance of the proposed change point estimator is evaluated in terms of accuracy and precision in comparison with the change point estimators developed under the assumptions of a step shift and drift. Finally, a numerical example is presented to show the application of the proposed change point estimator.

Keywords Statistical process control · Change point · Monotonic · Multivariate normal process · Maximum likelihood estimator

1 Introduction

Control charts are one of the most attractive and practical tools for process engineers to monitor the process. A process is in control as long as the points plot within the control limits, but if a point plots outside the control limits the process is out of control. To find the root cause(s), the process engineer should study the checklists from the last to the first one to find the best root. Therefore, knowing the time of change even approximately could be really helpful to save the time, energy and consequently the cost spent to find the disrupting noise.

Shewhart, cumulative sum (CUSUM), and exponentially weighted moving average (EWMA) control charts are widely used in monitoring univariate processes. CUSUM and EWMA control charts are built-in estimators which can be used to identify the time of change after the control chart gives an out-of-control signal while there is no such ability for Shewhart control charts. Many studies have been done to extract an effective estimator to identify the time of change in the parameters of a univariate quality characteristic.

On the other hand, advent of multivariate monitoring issue, opened attractive area of research that motivated many of specialists to do valuable studies especially in designing control charts such as χ^2,T^2, multivariate CUSUM and multivariate EWMA. These control charts are introduced to eliminate the limitations of multivariate processes monitoring.

As mentioned before, considerable studies have been carried out to identify the time of change. MLE as one of the most efficient techniques has attracted the attention of many researchers in the field of statistical process control. Amiri and Allahyari [2] provided a comprehensive review in the literature of change point estimation. They classified the papers based on seven indicators. One of them is the type of disturbance. They point out there are commonly three types of changes as: step shift, linear trend, and monotonic changes. In the single step change, the underlying parameter shifts once from the in-control value and levels off on new one until the control chart gives an OC alarm. Linear trend is another type of disruption in which the in-control parameter changes linearly

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Table 1 the IC mean vector and covariance matrix for 2 and 4 quality characteristics

p	μ_0	Σ_0
2	$(98\ 109)'$	$\begin{bmatrix} 4 & 1.68 \\ 1.68 & 16 \end{bmatrix}$
4	$(109\ 56\ 48\ 39)'$	$\begin{bmatrix} 1 & 0.49 & 0.56 & 2.13 \\ 0.49 & 1 & 1.16 & 5.03 \\ 0.56 & 1.16 & 16 & 6.07 \\ 2.13 & 5.03 & 6.07 & 36 \end{bmatrix}$

after the change point. The third type of change is more general than the step shift and drift, referred to as monotonic change.

Table 2 Accuracy of estimators used with step change in the mean vector after the χ^2 control chart gives an out-of-control signal for p=2 and p=4 (τ=30,N=10,000). Also standard error of estimations are shown in parenthesis

Number of variables	λ	$E(T)$	$\hat{\tau}$	$\hat{\tau}_{lt}$	$\hat{\tau}_{sc}$
p=2	0.5	231.45	39.53	17.45	36.88
			(0.31)	(0.30)	(0.20)
	0.75	149.99	26.51	11.94	31.72
			(0.13)	(0.24)	(0.14)
	1	99.89	25.17	13.41	30.61
			(0.09)	(0.15)	(0.06)
	1.5	53.72	24.88	20.82	30.17
			(0.08)	(0.09)	(0.03)
	2	39.30	24.51	26.22	30.02
			(0.09)	(0.06)	(0.02)
	2.5	34.57	24.62	28.27	29.84
			(0.07)	(0.03)	(0.02)
	3	32.58	24.61	29.13	29.84
			(0.06)	(0.02)	(0.01)
p=4	0.5	274.22	38.43	19.22	37.60
			(0.43)	(0.35)	(0.23)
	0.75	193.24	30.33	10.75	32.08
			(0.19)	(0.18)	(0.10)
	1	131.37	24.68	10.70	30.78
			(0.12)	(0.14)	(0.06)
	1.5	67.93	22.56	16.69	30.12
			(0.10)	(0.11)	(0.03)
	2	44.91	22.14	23.79	30.03
			(0.10)	(0.07)	(0.02)
	2.5	37.01	21.72	27.24	29.95
			(0.10)	(0.04)	(0.02)
	3	33.63	21.54	28.73	29.90
			(0.10)	(0.02)	(0.01)

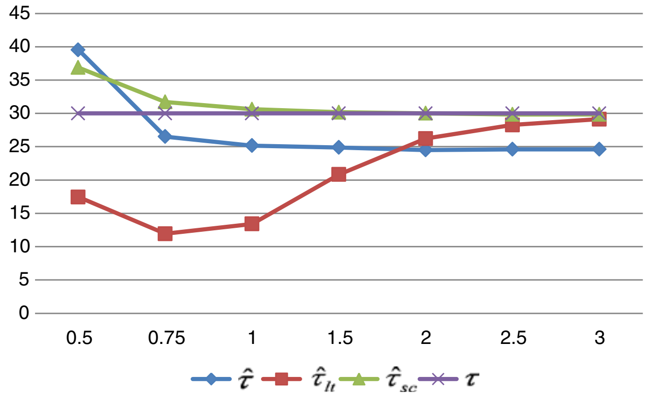

Fig. 1 Estimated accuracy of estimators under step change in the mean vector for different values of λ following the control chart gives an OC signal (p=2,τ=30 and N=10,000)

The main assumption in monotonic shift is that all of the disruptions types are able to take place but in a same direction. Isotonic shifts are increasingly monotonic shifts while antitonic shifts are decreasing.

There are many researches in the literature of change point estimation even though most of them are about the univariate change point estimators. Samuel et al. [13] proposed a MLE for estimating the time of a single step change in the Shewhart $\overline{X}$ control chart. Perry et al. [11] appraised the performance of MLE in estimating the time of change of the process nonconforming fraction with a monotonic change. They also compared their proposed estimator with the estimator proposed by Pignatiello and Samuel [12]. Noorossana and Shadman [9] studied the MLE for monotonic changes in the mean of a univariate normal process. They also compared performance of their proposed estimator with the estimators designed by Samuel et al. [13] and Perry and Pignatiello [10]. To have more specific classification of researches in this area see Amiri and Allahyari [2].

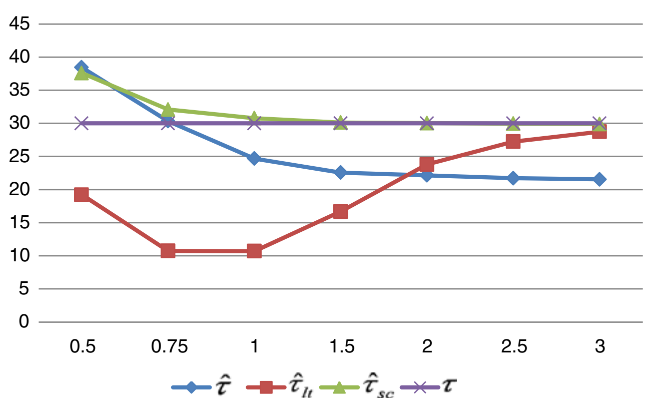

Fig. 2 Estimated accuracy of estimators under step change in the mean vector for different values of λ following the control chart gives an OC signal (p=4,τ=30 and N=10,000)

For multivariate processes in Phase I, Sullivan and Woodall [14] used likelihood ratio test to detect multiple change points in the process parameters with individual observations. Sullivan [15] used a clustering approach to find the number of changes and their location in a series of individual observations. He showed the proposed algorithm is considerably powerful than the X-chart and CUSUM in detection and estimation of disruptions when there are multiple shifts and/or outliers. Also, Zamba and Hawkins [16] proposed a change point model and used generalized likelihood ratio test to detect and estimate changes in the mean vector and/or covariance matrix. In Phase II, Nedumaran and Pignatiello [6] suggested a MLE for estimating the time of a step change in the mean vector of multivariate normal processes and assessed their proposed estimator performance with two criterions of accuracy and precision. Movaffagh and Amiri (submitted) designed a MLE when the elements of the mean vector of a multivariate Normal process changes linearly. They made a collation between the performance of their proposed MLE with the MLE suggested by Nedumaran and Pignatiello [6] and showed that their proposed method outperforms the MLE derived for step changes when a drift change occurs. Also, Niaki and Khedmati [7] used the MLE to estimate the time of step change in the mean vector of a multivariate Poisson process. At first, they used a transformation method to remove inherent skewness of Poisson process to form the data as multivariate normal and next applied another transformation technique to eliminate the correlation between attributes. The derivations of MLE are applied on transformed data. Accuracy and precision of their proposed method are evaluated through variety of numerical examples. In another study, Niaki and Khedmati [8] used the MLE for finding the time of change in the mean vector of a multivariate Poisson process when it changes linearly. Also they compared their proposed method with the MLE designed for the step change in the attribute control charts and showed that their suggested estimator outperforms the step change estimator.

For estimating the time of changes in covariance matrix, Doğu and Kocakoç [4] derived a MLE for finding the time of a step change in the generalized variance when |S|-chart is applied to monitor the dispersion of a multivariate normal process. They appraised the performance of their proposed method with some numerical simulations. Also Doğu and Kocakoç [5] used the maximum likelihood estimator for estimating the time of step change that is

Table 3 Estimated precision of estimators under step change in the mean vector for different values of λ after the χ^2 control chart gives an OC signal for p=2, τ=30 and N=10,000 (precision of the MLE designed for linear trend is shown in parenthesis and precision of the MLE designed for step change is depicted in bracket)

	λ=0.5	λ=0.75	λ=1	λ=1.5	λ=2	λ=2.5	λ=3
$\widehat{P}(\lvert\widehat{\tau}-\tau\rvert=0)$	0.08	0.16	0.22	0.28	0.29	0.30	0.31
	(0.01)	(0.02)	(0.04)	(0.11)	(0.25)	(0.41)	(0.58)
	[0.08]	[0.16]	[0.26]	[0.44]	[0.60]	[0.73]	[0.83]
$\widehat{P}(\lvert\widehat{\tau}-\tau\rvert\leq 1)$	0.18	0.33	0.40	0.45	0.44	0.44	0.44
	(0.02)	(0.04)	(0.07)	(0.21)	(0.42)	(0.64)	(0.80)
	[0.08]	[0.16]	[0.26]	[0.44]	[0.60]	[0.73]	[0.83]
$\widehat{P}(\lvert\widehat{\tau}-\tau\rvert\leq 2)$	0.26	0.43	0.50	0.54	0.54	0.53	0.53
	(0.02)	(0.04)	(0.07)	(0.21)	(0.42)	(0.64)	(0.80)
	[0.26]	[0.44]	[0.60]	[0.80]	[0.92]	[0.96]	[0.98]
$\widehat{P}(\lvert\widehat{\tau}-\tau\rvert\leq 3)$	0.33	0.52	0.57	0.61	0.60	0.59	0.60
	(0.04)	(0.07)	(0.14)	(0.36)	(0.63)	(0.83)	(0.94)
	[0.33]	[0.52]	[0.68]	[0.87]	[0.96]	[0.98]	[0.98]
$\widehat{P}(\lvert\widehat{\tau}-\tau\rvert\leq 4)$	0.38	0.58	0.62	0.66	0.65	0.65	0.65
	(0.04)	(0.09)	(0.17)	(0.41)	(0.70)	(0.88)	(0.97)
	[0.38]	[0.58]	[0.75]	[0.91]	[0.98]	[0.99]	[0.99]
$\widehat{P}(\lvert\widehat{\tau}-\tau\rvert\leq 5)$	0.43	0.64	0.67	0.70	0.69	(0.69)	0.69
	(0.05)	(0.11)	(0.20)	(0.46)	(0.74)	(0.92)	(0.98)
	[0.43]	[0.64]	[0.80]	[0.94]	[0.99]	[0.99]	[0.99]
$\widehat{P}(\lvert\widehat{\tau}-\tau\rvert\leq 10)$	0.60	0.79	0.81	0.83	0.81	0.82	0.82
	(0.09)	(0.17)	(0.29)	(0.61)	(0.88)	(0.98)	(0.99)
	[0.60]	[0.80]	[0.92]	[0.99]	[1.00]	[0.99]	[1.00]
$\widehat{P}(\lvert\widehat{\tau}-\tau\rvert\leq 15)$	0.69	0.87	0.88	0.89	0.88	0.89	0.89
	(0.14)	(0.26)	(0.41)	(0.75)	(0.96)	(0.99)	(1.00)
	[0.71]	[0.89]	[0.97]	[0.99]	[1.00]	[1.00]	[1.00]

imposed to the mean vector and covariance matrix simultaneously when χ^2 and $|S|$ control charts are used to monitor mean vector and dispersion of a multivariate normal process.

To the best of authors' knowledge there is no study on estimating the time of monotonic changes in the mean vector of multivariate normal processes. Since monotonic changes include more general aspects of changes, the estimation of the time of changes in presence of monotonic changes would be interesting. In this paper, we derived a MLE for estimating the time of change in the mean vector of a multivariate normal process when the pattern of change obeys from the monotonic model in Phase II.

In the next section, behavior of process under the isotonic changes is discussed and corresponding derivations is extracted. Through Section 3, performance of proposed MLE, MLE designed for linear trend and the MLE designed for step change is compared for two and four number of quality characteristics. A numerical example is given in Section 4 to show the application as well as the accuracy of the proposed change point estimator. Our concluding remarks are given in the final section.

2 Process model and derivation of the MLE

Let consider the $p\times 1$ vector of $\mathbf{x}_{ij}=(x_{ij1},x_{ij2},\ldots,x_{ijp})'$ as a representative of p quality characteristics on j^{th} $(j=1,2,\ldots,n)$ observation in ith subgroup. Moreover, we assume that the observations are independent and identically distributed and sampled from a multivariate normal process with in-control mean vector of μ_0 and covariance matrix of Σ_0. An appropriate control chart to monitor p variate normal process in Phase II is the χ^2 control chart. The ith statistic is calculated as follows:

$$\chi_i^2 = \left(\overline{\mathbf{x}}_i - \mu_0\right)' \boldsymbol{\Sigma}_0^{-1} \left(\overline{x}_i - \mu_0\right), \tag{1}$$

Where

$$\overline{\mathbf{x}}_i = \left(\frac{\sum_j x_{ij1}}{n}, \frac{\sum_j x_{ij2}}{n}, \ldots, \frac{\sum_j x_{ijp}}{n} \right)', \tag{2}$$

Table 4 Estimated precision of estimators under step change in the mean vector for different values of λ after the χ^2 control chart gives an OC signal for $p=4, \tau=30$ and $N=10{,}000$ (precision of the MLE designed for linear trend is shown in parenthesis and precision of the MLE designed for step change is depicted in bracket)

	λ=0.5	λ=0.75	λ=1	λ=1.5	λ=2	λ=2.5	λ=3
$\widehat{P}(\lvert\widehat{\tau}-\tau\rvert=0)$	0.07	0.08	0.13	0.18	0.22	0.22	0.23
	(0.01)	(0.01)	(0.02)	(0.07)	(0.17)	(0.31)	(0.48)
	[0.08]	[0.15]	[0.25]	[0.43]	[0.59]	[0.71]	[0.81]
$\widehat{P}(\lvert\widehat{\tau}-\tau\rvert\leq 1)$	0.16	0.19	0.27	0.33	0.36	0.35	0.35
	(0.01)	(0.02)	(0.05)	(0.14)	(0.31)	(0.50)	(0.71)
	[0.17]	[0.32]	[0.45]	[0.68]	[0.82]	[0.90]	[0.95]
$\widehat{P}(\lvert\widehat{\tau}-\tau\rvert\leq 2)$	0.23	0.27	0.37	0.43	0.44	0.43	0.42
	(0.02)	(0.04)	(0.07)	(0.19)	(0.40)	(0.63)	(0.83)
	[0.25]	[0.43]	[0.58]	[0.80]	[0.91]	[0.96]	[0.98]
$\widehat{P}(\lvert\widehat{\tau}-\tau\rvert\leq 3)$	0.28	0.34	0.44	0.49	0.50	0.49	0.48
	(0.02)	(0.05)	(0.09)	(0.24)	(0.48)	(0.72)	(0.89)
	[0.31]	[0.51]	[0.67]	[0.87]	[0.95]	[0.98]	[0.99]
$\widehat{P}(\lvert\widehat{\tau}-\tau\rvert\leq 4)$	0.33	0.40	0.50	0.55	0.55	0.54	0.53
	(0.03)	(0.06)	(0.11)	(0.28)	(0.54)	(0.78)	(0.93)
	[0.36]	[0.58]	[0.74]	[0.91]	[0.97]	[0.99]	[0.99]
$\widehat{P}(\lvert\widehat{\tau}-\tau\rvert\leq 5)$	0.37	0.45	0.55	0.59	0.59	0.57	0.56
	(0.04)	(0.07)	(0.13)	(0.32)	(0.60)	(0.83)	(0.95)
	[0.41]	[0.64]	[0.78]	[0.94]	[0.98]	[0.99]	[0.99]
$\widehat{P}(\lvert\widehat{\tau}-\tau\rvert\leq 10)$	0.52	0.61	0.71	0.71	0.71	0.70	0.69
	(0.07)	(0.13)	(0.22)	(0.47)	(0.78)	(0.95)	(0.99)
	[0.57]	[0.79]	[0.90]	[0.99]	[1.00]	[1.00]	[1.00]
$\widehat{P}(\lvert\widehat{\tau}-\tau\rvert\leq 15)$	0.61	0.71	0.79	0.79	0.78	0.77	0.77
	(0.11)	(0.19)	(0.29)	(0.59)	(0.88)	(0.98)	(1.00)
	[0.68]	[0.88]	[0.96]	[1.00]	[1.00]	[1.00]	[1.00]

The upper control limit for the statistic in Eq. (1) is computed as follows:

$$\mathrm{UCL} = \chi^2_{p,\alpha}, \quad (3)$$

where $\chi^2_{\alpha,p}$ is 100(1−α)percentile of chi-square distribution with p degrees of freedom. If the statistic in Eq. (1) exceeds the UCL in Eq. (3), the χ^2 control chart will notify the OC state of process. Note that the corresponding time of observation which is plotted above the UCL is not necessarily the real time of change.

Table 5 Accuracy of estimators under linear trend shifts in the mean vector after the χ^2 control chart gives an out-of-control signal for p=2 and p=4 (τ=30,N=10,000)

Number of variables	β	$E(T)$	$\hat{\tau}$	$\hat{\tau}_{lt}$	$\hat{\tau}_{sc}$
p=2	(0.01 0.01)′	121.95	54.41	40.65	67.22
			(0.18)	(0.24)	(0.21)
	(0.03 0.03)′	73.83	35.87	34.67	47.27
			(0.11)	(0.12)	(0.10)
	(0.07 0.07)′	53.88	29.68	32.12	38.78
			(0.09)	(0.07)	(0.06)
	(0.1 0.1)′	48.18	28.29	31.66	36.61
			(0.08)	(0.06)	(0.05)
	(0.5 0.5)′	35.34	25.06	30.27	31.50
			(0.08)	(0.02)	(0.02)
	(1 1)′	33.15	24.61	29.97	30.68
			(0.08)	(0.02)	(0.02)
	(1.5 1.5)′	32.34	24.64	29.87	30.40
			(0.08)	(0.01)	(0.02)
	(2.5 2.5)′	31.65	24.62	29.80	30.20
			(0.07)	(0.01)	(0.01)
p=4	(0.01 0.01 0.01 0.01)′	125.93	50.83	39.27	67.84
			(0.19)	(0.22)	(0.20)
	(0.03 0.03 0.03 0.03)′	75.50	32.20	33.56	47.17
			(0.13)	(0.11)	(0.09)
	(0.07 0.07 0.07 0.07)′	54.33	26.67	32.11	39.01
			(0.11)	(0.07)	(0.05)
	(0.1 0.1 0.1 0.1)′	48.74	24.95	31.42	36.73
			(0.11)	(0.05)	(0.05)
	(0.5 0.5 0.5 0.5)′	35.35	21.82	30.27	31.52
			(0.10)	(0.02)	(0.02)
	(1 1 1 1)′	33.13	21.39	29.99	30.66
			(0.10)	(0.02)	(0.02)
	(1.5 1.5 1.5 1.5)′	32.31	21.30	29.88	30.40
			(0.10)	(0.01)	(0.01)
	(2.5 2.5 2.5 2.5)′	31.88	21.27	29.82	30.25
			(0.10)	(0.01)	(0.01)

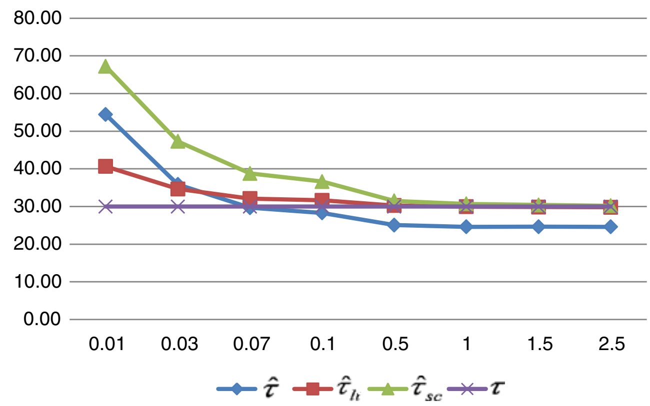

Fig. 3 Estimated accuracy of estimators under linear trend shifts in the mean vector following the control chart gives an OC signal (p=2,τ=30 and N=10,000)

In this section, effect of monotonic changes in the mean vector of a multivariate process is studied and corresponding equations of change and the appropriate MLE to identify the real time of change is extracted. Recall from the previous section the monotonic changes include a series of changes which have a same direction (increasing or decreasing) but the exact type of changes are not clear.

Without loss of generality we consider that the types of changes are isotonic. So, the samples are gathered from an IC process with known mean vector of $\mathbf{\mu}_0$ but after an unknown time τ the first disruption is introduced to the mean vector and changes it to another vector. The process will be in the out-of-control state until the χ^2 control chart gives a genuine signal at the time T. Since we assumed that the pattern of changes are isotonic, for $i=\tau+1,\tau+2,\ldots,T$ samples are collected from a multivariate normal process with the mean vector of $\mathbf{\mu}_i$, where $\mathbf{\mu}_{\tau+1}>\mathbf{\mu}_0$ and $\mathbf{\mu}_i\geq\mathbf{\mu}_{i-1}$.

We denote the maximum likelihood estimation of the first change point by $\hat{\tau}$. To extract the derivations we start with

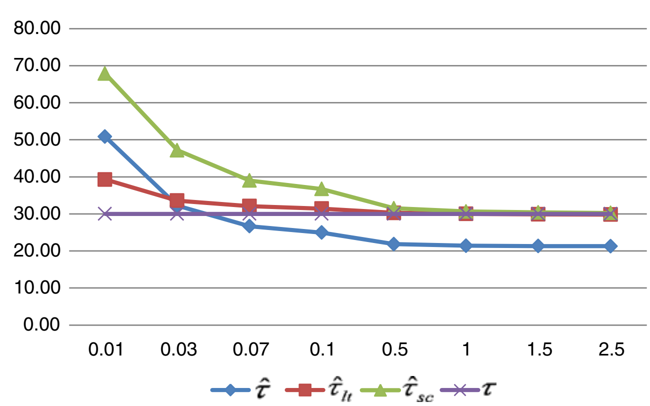

Fig. 4 Estimated accuracy of estimators under linear trend shifts in the mean vector following the control chart gives an OC signal (p=4,τ=30 and N=10,000)

the likelihood function of multivariate normal distribution which is given in Eq. (4):

$$L(\tau, \boldsymbol{\mu}_{T-t}|\mathbf{x}, \boldsymbol{\mu}_0) = \prod_{i=1}^{\tau} \frac{1}{(2\pi)^{p/2}\left|\boldsymbol{\Sigma}_0/n\right|^{1/2}} e^{-\frac{n}{2}\left[\left(\overline{\mathbf{x}}_i - \boldsymbol{\mu}_0\right)' \boldsymbol{\Sigma}_0^{-1}\left(\overline{\mathbf{x}}_i - \boldsymbol{\mu}_0\right)\right]} \prod_{i=\tau+1}^{T} \frac{1}{(2\pi)^{p/2}\left|\boldsymbol{\Sigma}_0/n\right|^{1/2}} e^{-\frac{n}{2}\left[\left(\overline{\mathbf{x}}_i - \boldsymbol{\mu}_i\right)' \boldsymbol{\Sigma}_0^{-1}\left(\overline{\mathbf{x}}_i - \boldsymbol{\mu}_i\right)\right]}, \quad (4)$$

where, $\boldsymbol{\mu}_{T-\tau}=[\boldsymbol{\mu}_{\tau+1},\boldsymbol{\mu}_{\tau+2},\ldots,\boldsymbol{\mu}_T]$ is the vector of OC mean vectors and $\mathbf{x}$ denotes the given vector of observations. Taking the logarithm from the likelihood function in Eq. (4), we obtain:

$$\ln L(\tau, \boldsymbol{\mu}_{T-\tau}|\mathbf{x}, \boldsymbol{\mu}_0) = K - \frac{n}{2}\sum_{i=1}^{\tau}\left[\left(\overline{\mathbf{x}}_i - \boldsymbol{\mu}_0\right)' \boldsymbol{\Sigma}_0^{-1}\left(\overline{\mathbf{x}}_i - \boldsymbol{\mu}_0\right)\right] - \frac{n}{2}\sum_{i=\tau+1}^{T}\left[\left(\overline{\mathbf{x}}_i - \boldsymbol{\mu}_i\right)' \boldsymbol{\Sigma}_0^{-1}\left(\overline{\mathbf{x}}_i - \boldsymbol{\mu}_i\right)\right], \quad (5)$$

where, K is a constant. Since the values of τ and $\boldsymbol{\mu}_i$ for $i=\tau+1,\tau+2,\ldots,T$ are unknown, they must be replaced by approximated values. Hence, $\widehat{\tau}$ will be the MLE of the first change point if it maximizes the following expression:

$$\widehat{\tau} = \underset{0<t<T}{\operatorname{argmax}}\left(-\left\{\sum_{i=1}^{t}\left[\left(\overline{\mathbf{x}}_i - \boldsymbol{\mu}_0\right)' \boldsymbol{\Sigma}_0^{-1}\left(\overline{\mathbf{x}}_i - \boldsymbol{\mu}_0\right)\right] + \sum_{i=t+1}^{T}\left[\left(\overline{\mathbf{x}}_i - \widehat{\boldsymbol{\mu}}_i\right)' \boldsymbol{\Sigma}_0^{-1}\left(\overline{\mathbf{x}}_i - \widehat{\boldsymbol{\mu}}_i\right)\right]\right\}\right). \quad (6)$$

As aforementioned, the mean vector in out-of-control state should be approximated. The second expression of Eq. (6) includes $\boldsymbol{\mu}_i$ with the assumption of $\boldsymbol{\mu}_i \geq \boldsymbol{\mu}_{i-1}$ that emphasizes the mean vectors in OC state have a non-decreasing pattern.

At first, it should be explained how a non-decreasing manner takes place in the mean vector because comparison between vectors is not exactly the same as scalars. One can use just norm of vectors to make a collation on their greatness, but it is not enough to ensure a vector is greater than another one. It is obvious that if each element of a vector be greater than or equal to corresponding element in the other vector, the first vector will have a greater norm than the

Table 6 Accuracy of estimators under linear trend shifts in the mean vector when one or some of the variables changes after the χ^2 control chart gives an out-of-control signal ($p=2, \tau=30, N=10{,}000$)

β	$E(T)$	$\widehat{\tau}$	$\widehat{\tau}_{lt}$	$\widehat{\tau}_{sc}$
(0.01 0)′	121.777	60.64	42.19	67.87
(0.03 0)′	74.4209	37.94	34.60	47.33
(0.1 0)′	48.5231	28.92	31.59	36.72
(1 0)′	33.1926	24.64	29.99	30.70
(0 0.01)′	169.66	90.91	52.01	89.20
(0 0.03)′	102.194	49.70	38.06	58.92
(0 0.1)′	60.8642	33.03	33.04	41.78
(0 1)′	35.4076	25.26	30.24	31.51
(0.01 0 0 0)′	91.06	42.11	35.18	53.32
(0 0 0.07 0)′	74.20	34.29	33.52	46.68
(0 0 0 0.1)′	60.34	31.83	32.23	41.14
(0 0.03 0 0)′	48.25	27.89	31.39	36.52
(0.01 0.03 0 0)′	49.05	27.66	31.44	36.85
(0.03 0 0.07 0.01)′	40.17	24.13	30.65	33.35
(0 0.07 0 1)′	36.88	22.19	30.43	32.12
(0 0 0.5 1)′	34.84	22.10	30.18	31.34

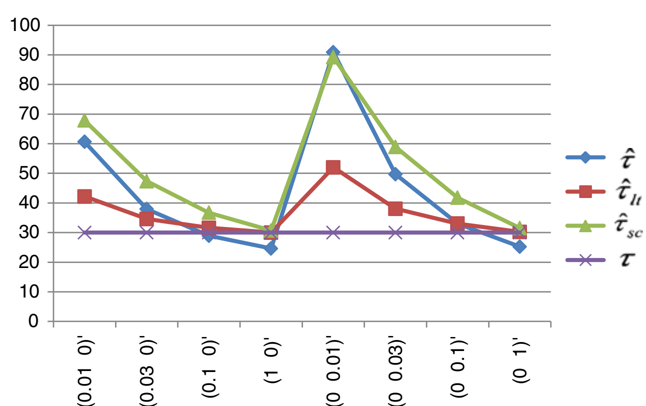

Fig. 5 Estimated accuracy of estimators under linear trend shifts in the mean vector following the control chart gives an OC signal ($p=2, \tau=30$ and $N=10{,}000$)

second one. Without loss of generality, to gain OC vectors that satisfy the constraint of $\mu_i \geq \mu_{i-1}$, each element of OC vectors is set to be greater than or equal to each corresponding element in the IC vector. We use the non-central parameter of χ^2 distribution namely λ which is defined in Eq. (7) to make a sensible indicator of change for single and multiple step changes.

$$\lambda = \sqrt{n(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0)' \boldsymbol{\Sigma}_0^{-1} (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0)} \tag{7}$$

To obtain initial approximation of non-decreasing changes in the mean vector of process we define a set of $k \times 1$ column vectors which are denoted as below:

$$\widetilde{\mu}_{ik} = \begin{cases} \overline{x}_{ik} & if \quad \overline{x}_{ik} > \mu_{0k} \\ \mu_{0k} & if \quad \overline{x}_{ik} \leq \mu_{0k} \end{cases} \tag{8}$$

for $i = t+1, t+2, \ldots, T$ and $k = 1, 2, \ldots, p$,

where $\overline{x}_{ik}$ and μ_{0k} are the kth element of ith sample mean vector and IC mean vector of the process, respectively. Equation (8) constructs a primary approximation for modeling the isotonic pattern by adjusting each element of OC vectors. Although Eq. (8) is applied to provide a primary approximation of OC mean vectors in the presence of isotonic change, it is not precise enough. To gain a more precise estimation of OC mean vectors, we use the mathematical programming model in Eq. (9).

By solving the strictly convex quadratic programming model in Eq. (9), the elements of OC mean vector for kth variable are achieved. By repeating the solving procedure for all of the variables, a set of OC mean vectors that has least distance from the initial approximation by satisfying $\mu_i \geq \mu_{i-1}$ is obtained.

$$\min \sum_{i=1}^{T} \left(\widehat{\mu}_{ik} - \widetilde{\mu}_{ik} \right)^2 \quad st. \quad \widehat{\mu}_{ik} \geq \widehat{\mu}_{(i-1)k} \quad for\ i = 1, \ldots, T \tag{9}$$

To solve the quadratic model in Eq. (9), one can implement pooled-adjacent-violators (PAV) algorithm which can be considered as active set method. As mentioned in Best and Chakravarti [3], quadratic programming is the natural generalization of linear programming and active set methods are the natural generalization of simplex method. We used the steps of

Table 7 Estimated precision of estimators under linear trend in the mean vector for different vectors of slope after the χ^2 control chart gives an OC signal for p=2, τ=30 and N=10,000 (precision of $\widehat{\tau}_{lt}$ is shown in parenthesis and precision of $\widehat{\tau}_{sc}$ is depicted in bracket)

β	$\begin{pmatrix}0.01\\0.01\end{pmatrix}$	$\begin{pmatrix}0.03\\0.03\end{pmatrix}$	$\begin{pmatrix}0.07\\0.07\end{pmatrix}$	$\begin{pmatrix}0.1\\0.1\end{pmatrix}$	$\begin{pmatrix}0.5\\0.5\end{pmatrix}$	$\begin{pmatrix}1\\1\end{pmatrix}$	$\begin{pmatrix}1.5\\1.5\end{pmatrix}$	$\begin{pmatrix}2.5\\2.5\end{pmatrix}$
$\widehat{P}(\lvert\widehat{\tau}-\tau\rvert = 0)$	0.01	0.03	0.05	0.07	0.18	0.25	0.29	0.31
	(0.02)	(0.04)	(0.07)	(0.08)	(0.24)	(0.42)	(0.57)	(0.78)
	[0.01]	[0.01]	[0.02]	[0.03]	[0.13]	[0.28]	[0.42]	[0.63]
$\widehat{P}(\lvert\widehat{\tau}-\tau\rvert \leq 1)$	0.03	0.09	0.16	0.21	0.42	0.45	0.45	0.45
	(0.06)	(0.12)	(0.20)	(0.24)	(0.60)	(0.85)	(0.92)	(0.96)
	[0.01]	[0.03]	[0.05]	[0.07]	[0.40]	[0.74]	[0.88]	[0.98]
$\widehat{P}(\lvert\widehat{\tau}-\tau\rvert \leq 2)$	0.04	0.14	0.26	0.34	0.55	0.53	0.54	0.53
	(0.10)	(0.19)	(0.31)	(0.38)	(0.82)	(0.95)	(0.97)	(0.98)
	[0.02]	[0.04]	[0.09]	[0.13]	[0.69]	[0.95]	[0.98]	[0.99]
$\widehat{P}(\lvert\widehat{\tau}-\tau\rvert \leq 3)$	0.06	0.19	0.36	0.46	0.61	0.60	0.60	0.60
	(0.13)	(0.25)	(0.41)	(0.49)	(0.93)	(0.97)	(0.98)	(0.99)
	[0.03]	[0.06]	[0.13]	[0.20]	[0.89]	[0.98]	[0.99]	[0.99]
$\widehat{P}(\lvert\widehat{\tau}-\tau\rvert \leq 4)$	0.08	0.24	0.46	0.56	0.66	0.65	0.65	0.65
	(0.17)	(0.32)	(0.50)	(0.58)	(0.97)	(0.98)	(0.99)	(0.99)
	[0.04]	[0.08]	[0.18]	[0.27]	[0.97]	[0.99]	[0.99]	[0.99]
$\widehat{P}(\lvert\widehat{\tau}-\tau\rvert \leq 5)$	0.10	0.30	0.54	0.65	0.70	0.68	0.69	0.69
	(0.21)	(0.38)	(0.58)	(0.66)	(0.98)	(0.99)	(0.99)	(0.99)
	[0.05]	[0.10]	[0.24]	[0.36]	[0.99]	[0.99]	[0.99]	[1.00]
$\widehat{P}(\lvert\widehat{\tau}-\tau\rvert \leq 10)$	0.18	0.56	0.83	0.86	0.82	0.82	0.82	0.82
	(0.36)	(0.61)	(0.84)	(0.91)	(0.99)	(1.00)	(1.00)	(1.00)
	[0.10]	[0.23]	[0.60]	[0.80]	[0.99]	[1.00]	[1.00]	[1.00]
$\widehat{P}(\lvert\widehat{\tau}-\tau\rvert \leq 15)$	0.28	0.77	0.92	0.91	0.88	0.88	0.88	0.89
	(0.49)	(0.76)	(0.95)	(0.98)	(1.00)	(1.00)	(1.00)	(1.00)
	[0.14]	[0.41]	[0.88]	[0.99]	[1.00]	[1.00]	[1.00]	[1.00]

PAV algorithm which its complexity is evaluated by Best and Chakravarti [3] in solving isotonic regression problems.

3 Performance evaluation

In this section, the performance of our proposed MLE ($\widehat{\tau}$) is compared with the MLE designed for linear trend derived by Movaffagh and Amiri (submitted) and the MLE proposed by Nedumaran and Pignatiello [6] which are denoted by $\widehat{\tau}_{lt}$ and $\widehat{\tau}_{sc}$, respectively. To have a collation, we study performance of the estimators for 3 different classes of changes as (1) a single step shift, (2) a linear trend type of shift and (3) multiple step changes in the process mean vector.

During the simulation process, the data are generated from a process with IC parameters for $1 \leq t \leq \tau$. To eliminate the false alarm of control chart if a chi-square statistic plots above the UCL, it is replaced by another. This procedure is iterated until the sample statistic lies lower than the UCL. In $t=\tau+1$, the mean vector is altered based on the type of the change and its magnitude. Hence, for $t \geq \tau+1$samples are generated from an out-of-control process until the χ^2 control chart gives an out-of-control signal. As mentioned in previous section, for step and monotonic changes we set the magnitude of λ and find the corresponding OC mean vector that gives the specified value of λ which satisfies $\mathbf{\mu}_i \geq \mathbf{\mu}_{i-1}$ for $i \geq \tau+1$. However, for linear trend shifts we use the model in Eq. (10) to change the mean vector.

3.1 Evaluating change point estimators under a step change interruption

In this section, the performance of the proposed estimator is appraised when a step change occurred in the mean vector of a multivariate normal process. Also, we compare the results of our estimator with the MLE designed for linear trend and step change. Without loss of generality, we consider a multivariate normal process with $p=2$and $p=4$ quality characteristics and

Table 8 Estimated precision of estimators under linear trend shift in the mean vector for different vectors of slope after the χ^2 control chart gives an OC signal for $p=4, \tau=30$ and $N=10{,}000$ (precision of $\widehat{\tau}_{lt}$ is shown in parenthesis and precision of $\widehat{\tau}_{sc}$ is depicted in bracket)

β	$\begin{pmatrix}0.01\\0.01\\0.01\\0.01\end{pmatrix}$	$\begin{pmatrix}0.03\\0.03\\0.03\\0.03\end{pmatrix}$	$\begin{pmatrix}0.07\\0.07\\0.07\\0.07\end{pmatrix}$	$\begin{pmatrix}0.1\\0.1\\0.1\\0.1\end{pmatrix}$	$\begin{pmatrix}0.5\\0.5\\0.5\\0.5\end{pmatrix}$	$\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$	$\begin{pmatrix}1.5\\1.5\\1.5\\1.5\end{pmatrix}$	$\begin{pmatrix}2.5\\2.5\\2.5\\2.5\end{pmatrix}$
$\widehat{P}(\lvert\widehat{\tau}-\tau\rvert=0)$	0.01	0.03	0.05	0.07	0.15	0.20	0.21	0.21
	(0.02)	(0.05)	(0.08)	(0.10)	(0.27)	(0.46)	(0.62)	(0.73)
	[0.00]	[0.01]	[0.01]	[0.02]	[0.12]	[0.29]	[0.45]	[0.59]
$\widehat{P}(\lvert\widehat{\tau}-\tau\rvert\leq 1)$	0.03	0.08	0.15	0.20	0.33	0.34	0.34	0.33
	(0.07)	(0.14)	(0.22)	(0.27)	(0.64)	(0.88)	(0.94)	(0.95)
	[0.01]	[0.02]	[0.05]	[0.06]	[0.41]	[0.78]	[0.91]	[0.97]
$\widehat{P}(\lvert\widehat{\tau}-\tau\rvert\leq 2)$	0.05	0.14	0.25	0.31	0.43	0.42	0.41	0.40
	(0.12)	(0.22)	(0.35)	(0.42)	(0.85)	(0.96)	(0.97)	(0.98)
	[0.02]	[0.03]	[0.08]	[0.11]	[0.71]	[0.96]	[0.98]	[0.99]
$\widehat{P}(\lvert\widehat{\tau}-\tau\rvert\leq 3)$	0.06	0.19	0.35	0.42	0.49	0.48	0.47	0.46
	(0.16)	(0.29)	(0.46)	(0.55)	(0.95)	(0.98)	(0.99)	(0.99)
	[0.02]	[0.05]	[0.12]	[0.18]	[0.91]	[0.99]	[0.99]	[0.99]
$\widehat{P}(\lvert\widehat{\tau}-\tau\rvert\leq 4)$	0.08	0.25	0.44	0.51	0.53	0.52	0.51	0.51
	(0.19)	(0.37)	(0.56)	(0.65)	(0.98)	(0.99)	(0.99)	(0.99)
	[0.03]	[0.06]	[0.17]	[0.26]	[0.98]	[0.99]	[0.99]	[0.99]
$\widehat{P}(\lvert\widehat{\tau}-\tau\rvert\leq 5)$	0.10	0.30	0.52	0.58	0.57	0.56	0.55	0.55
	(0.23)	(0.43)	(0.64)	(0.72)	(0.99)	(0.99)	(0.99)	(1.00)
	[0.04]	[0.08]	[0.23]	[0.35]	[0.99]	[0.99]	[0.99]	[1.00]
$\widehat{P}(\lvert\widehat{\tau}-\tau\rvert\leq 10)$	0.20	0.57	0.76	0.76	0.70	0.69	0.69	0.69
	(0.41)	(0.68)	(0.86)	(0.93)	(1.00)	(1.00)	(1.00)	(1.00)
	[0.08]	[0.22]	[0.60]	[0.81]	[1.00]	[1.00]	[1.00]	[1.00]
$\widehat{P}(\lvert\widehat{\tau}-\tau\rvert\leq 15)$	0.30	0.76	0.83	0.81	0.77	0.77	0.76	0.77
	(0.55)	(0.82)	(0.96)	(0.99)	(1.00)	(1.00)	(1.00)	(1.00)
	[0.13]	[0.41]	[0.89]	[0.99]	[1.00]	[1.00]	[1.00]	[1.00]

the in-control mean vector of μ_0 and covariance matrix of Σ_0 as shown in Table 1.

We set the real change point to τ=30 for different magnitudes of change λ=0.5,0.75,1,1.5,2 and 2.5 and run 10,000 Mont Carlo simulations for each λ to calculate the performance indicators namely accuracy and precision. Average number of samples to receive an OC alarm by the χ^2 control chart, accuracy of estimators and standard error of estimations under different values of λ are shown in Table 2. In order to have a better view of comparisons, Figs. 1 and 2 are used to depict the accuracy of the estimators for different λ.

As Table 2 shows, when a step change disturbance is imposed to the mean vector of the process the step change MLE outperforms the other estimators. However, our designed estimator has a better performance than $\widehat{\tau}_{lt}$ when the magnitude of λ is less than 2 which implies that for small shift sizes, our estimator is considerably more accurate than the MLE designed for linear trend shifts.

Generally, when the type of change in the mean vector is a single step shift, the estimator designed for step change has the best performance. However, it should be considered that step change estimator is constructed based on the assumption

Table 9 Accuracy of estimators under three increasing step changes in the mean vector after the χ^2 control chart gives an out–of-control signal (τ_1=25, τ_2=35, τ_3=45, N=10000)

Number of variables	λ_1	λ_2	λ_3	$E(T)$	$\widehat{\tau}$	$\widehat{\tau}_{lt}$	$\widehat{\tau}_{sc}$
p=2	0.25	0.5	0.75	151.55	30.23	14.20	36.68
					(0.12)	(0.19)	(0.13)
	0.25	0.75	1.25	79.13	27.40	20.24	36.67
					(0.10)	(0.15)	(0.08)
	0.25	1	1.75	56.19	26.62	27.01	36.35
					(0.09)	(0.10)	(0.06)
	0.5	0.75	1	101.82	23.86	13.62	32.11
					(0.10)	(0.15)	(0.09)
	0.5	1	1.5	62.73	23.38	20.45	33.15
					(0.09)	(0.12)	(0.07)
	0.5	1.25	2	50.30	23.14	25.15	33.12
					(0.09)	(0.09)	(0.06)
	1	1.25	1.5	57.61	20.29	15.90	27.13
					(0.07)	(0.10)	(0.05)
	1	1.5	2	46.84	20.23	20.03	27.74
					(0.07)	(0.08)	(0.05)
	1	1.75	2.5	42.68	20.29	22.09	28.06
					(0.07)	(0.06)	(0.05)
p=4	0.25	0.5	0.75	195.60	26.62	12.73	36.74
					(0.14)	(0.20)	(0.13)
	0.25	0.75	1.25	101.29	23.77	16.15	37.07
					(0.12)	(0.15)	(0.08)
	0.25	1	1.75	64.74	22.98	23.96	36.85
					(0.11)	(0.12)	(0.06)
	0.5	0.75	1	133.57	20.40	11.11	32.29
					(0.11)	(0.15)	(0.10)
	0.5	1	1.5	77.20	19.89	17.04	33.89
					(0.11)	(0.13)	(0.07)
	0.5	1.25	2	56.22	19.81	23.36	33.86
					(0.10)	(0.10)	(0.06)
	1	1.25	1.5	72.09	17.11	12.98	27.44
					(0.09)	(0.11)	(0.05)
	1	1.5	2	53.04	17.04	18.31	28.26
					(0.09)	(0.09)	(0.05)
	1	1.75	2.5	46.18	16.96	21.34	28.73
					(0.09)	(0.07)	(0.05)

that the type of change is known, while our proposed estimator does not require the assumption of knowing the exact type of change.

Estimated precision of estimators are summarized in Tables 3 and 4 for two and four variables, respectively. It can be clearly seen that the precision of all of the estimators increases when the magnitude of shift grows up. Results reveal that for a specific neighborhood radius from the real change point our proposed estimator and the MLE designed for step change is very close to each other for $\lambda<2$and noticeably differs from the poor precision of the MLE designed for linear trend changes. However, for large sizes of λ, estimated precision of $\widehat{\tau}_{lt}$ improves. Moreover, by increasing the neighborhood radius, the estimated precision of $\widehat{\tau}$ and $\widehat{\tau}_{sc}$ rise faster than $\widehat{\tau}_{lt}$ for $\lambda<2$. However, for almost large values of λ, the MLE designed for drift shift shows sound precision.

3.2 Evaluating change point estimators under a drift disturbance

In this subsection, a linear trend interruption is imposed to the mean vector of the process. Assignable causes such as worker's exhaustion or tool's wear and so on can result in this kind of change in the process. Mathematical model of drift is given in Eq. (10).

$$\boldsymbol{\mu}_i = \boldsymbol{\mu}_0 + \boldsymbol{\beta}(i-\tau) \quad \text{for } i = \tau+1, \ldots\ , \tag{10}$$

where $\boldsymbol{\beta}$ is the vector of slopes and τ is the real time of change. Equation (10) indicates that before the real time of change samples are gathered from an IC process, but after the time τ the mean vector changes linearly. This manner is seen until the chi-square control chart gives an out-of-control signal.

We used the mean vector and covariance matrix which are mentioned in previous subsection in our simulation studies. The same as the step change, we set the real change point to τ=30 and use 10,000 independent Mont Carlo simulation runs

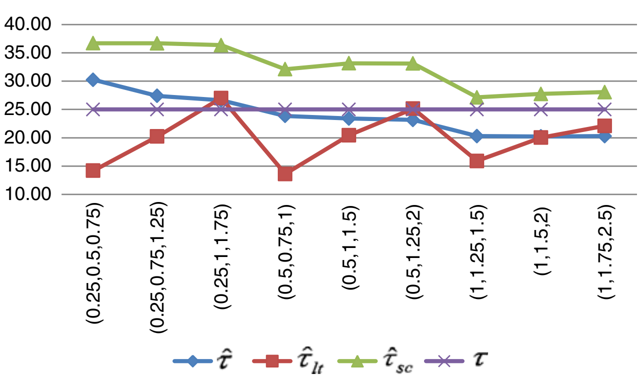

Fig. 6 Estimated accuracy of estimators under three increasing step changes in the mean vector following the control chart gives an out-of-control signal (p=2, τ_1=25, τ_2=35, τ_3=45, N=10,000)

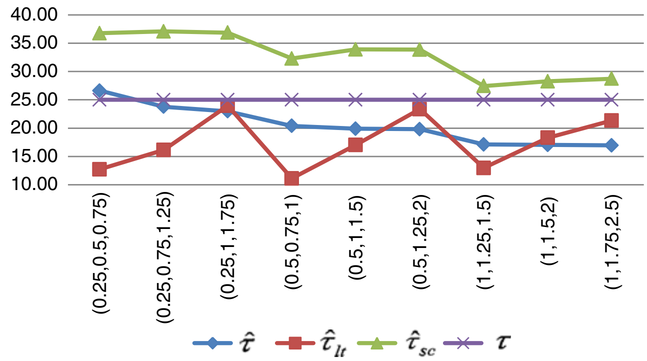

Fig. 7 Estimated accuracy of estimators under three increasing step changes in the mean vector following the control chart gives an out-of-control signal (p=4, τ_1=25, τ_2=35, τ_3=45, N=10,000)

for different vectors of slope to evaluate the performance of the estimators. Table 5, besides Figs. 3 and 4, describes the accuracy of estimators in presence of drift. For unequal magnitudes of elements in β, Table 6 and Fig. 5 are used to show the accuracy of estimators. Also, Tables 7 and 8 are used to make a collation between the estimators based on the precision criterion.

Table 10 Likelihood estimation of the first change point under three step changes after the χ^2 control chart gives an OC signal for stiffness and bending strength of lumbers

Sample number (i)	Stiffness ($\bar{x}_{i1}$)	Bending strength ($\bar{x}_{i2}$)	χ_i^2	LE	LE_{lt}	LE_{sc}
1	266.12	473.17	0.44	5.36	3.18	2.40
2	271.88	476.22	2.55	5.11	3.18	2.05
3	260.35	462.72	2.23	5.41	**3.21**	2.50
4	266.88	473.84	0.61	5.33	3.20	2.46
5	268.55	470.86	0.77	5.27	3.20	2.35
6	266.80	471.77	0.18	5.24	3.20	2.35
7	265.82	465.83	1.46	5.35	3.21	2.50
8	264.05	473.65	1.23	5.34	3.20	2.66
9	262.33	460.25	4.48	**5.65**	3.17	3.24
10	270.25	471.00	1.77	5.43	3.08	3.06
11	269.32	479.75	4.01	4.87	3.00	2.69
12	266.00	476.56	2.30	4.59	2.96	2.71
13	276.96	479.36	7.29	3.94	2.88	1.83
14	263.78	465.75	0.84	4.44	2.96	2.18
15	269.28	476.12	1.62	4.14	2.99	1.95
16	261.78	467.85	0.52	4.52	3.09	2.54
17	265.45	475.85	2.00	4.30	3.08	2.81
18	265.19	476.39	2.54	4.04	2.95	**3.36**
19	273.92	475.66	3.98	3.34	2.57	2.61
20	272.23	471.40	3.34	2.91	2.29	2.00
21	273.03	474.40	3.28	2.34	2.14	1.36
22	266.42	472.67	0.29	2.50	2.40	1.85
23	268.97	472.69	0.79	2.38	2.38	2.38
24	279.58	475.18	11.91			

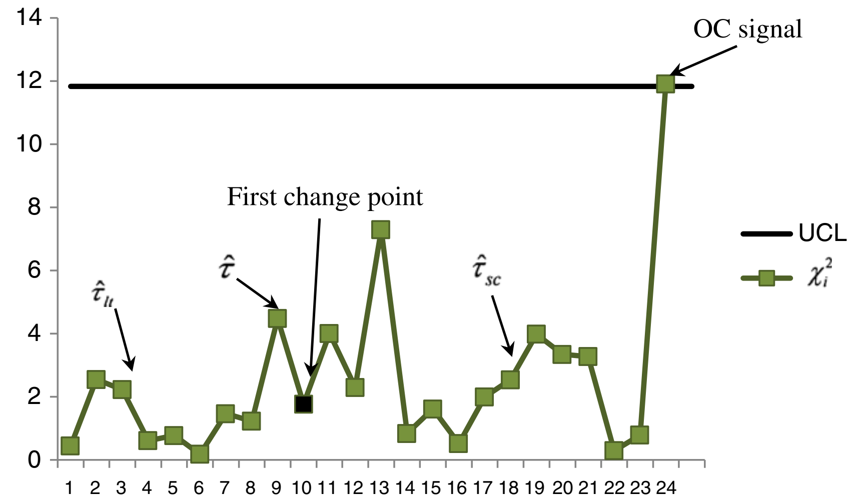

Fig. 8 Accuracy of estimators under three step changes after χ^2control chart gives a signal for lumber manufacturing process with two quality characteristics

The results from Table 5 show that when the type of change in the mean vector obeys from the model described in Eq. (10), our proposed estimator have a better performance than estimator of step change for relatively small change rates for both 2 and 4 quality characteristics. However, the estimator which is directly modeled for linear trend shifts outperforms the other estimators. This is due to the assumption of knowing change type during the construction of model of estimator in presence of linear trend shifts. For the slope vectors with magnitude of greater than 0.1 in each element, the performance of the MLE of step change shows more accurate estimation than the proposed estimator for monotonic changes.

3.3 Evaluating change point estimators under monotonic change

In this subsection, the performance of the estimators is reported when increasing multiple step changes take place in the mean vector of the process. To simulate the mentioned type of change we set a vector of the real time of multiple step changes as $\boldsymbol{\tau}$=(25 35 45) that means the first step change in the mean vector happens between samples 25 and 26, the second change takes place between samples 35 and 36 and the last ascending step change is occurred in the mean vector between samples 45 and 46 and the mean vector is remained constantly at the last level until the control chart gives an out-of-control signal. Similar to Section 3.1, we used the non-central parameter of λ to show the magnitude of change in the mean vector. For example, the first row of Table 9 denotes that after the first change points the out-of-control mean vector is changed such that the λ takes the value of 0.25. Also, after the second change point, the OC mean vector is determined such that λ to be equal to 0.5 and finally after the last change point the magnitude of λ is set to 0.75 by changing the mean vector.

As shown in Table 9, accuracy of estimators is highly affected by the magnitude of the change in the mean vector at the first change point. In other words, for almost small magnitudes of λ_1 (0.25 and 0.5) our proposed estimator considerably outperforms the other estimators, while for λ_1=1 the MLE designed for step change estimates the first change point more accurately. Also, behavior of $\widehat{\tau}_{lt}$ changes is not affected by increasing in λ_1. For a constant λ_1, if the gap between the magnitudes of λ_i for i=1,2,3 increases, our proposed estimator and the estimator proposed by Nedumaran and Pignatiello [6] almost level off while the MLE designed for drift shows an improving behavior. Figures 6 and 7 confirm this behavior of estimators.

4 Numerical example

In this section, we use the real case of lumber manufacturing which is exampled by Alt [1]. Two quality characteristics of stiffness and bending strength are used to monitor the process. Through the simulation, the subgroup size is set equal to n=5. Since the simulation is run in Phase II the mean vector and covariance matrix are assumed to be known based on the IC data set in Phase I as follows:

$$\boldsymbol{\mu}_0 = (266 \;\; 470)' \quad \boldsymbol{\Sigma}_0 = \begin{bmatrix} 100 & 66 \\ 66 & 121 \end{bmatrix},$$

so by setting the probability of type I error equal to 0.0027, the χ^2control chart with UCL $= \chi^2_{0.0027,2} = 11.8290$ is used to monitor the mean vector. Similar to Section 3.3, the real change points are set equal to $\boldsymbol{\tau}$=(10,15,20) and the mean vector is changed based on the values of 0.5, 1.25 and 2 for λ_1, λ_2 and λ_3, respectively. Sample mean for each quality characteristic, chi-square statistic for each sample and likelihood estimations of each estimator are summarized in different columns of Table 10 and denoted by LE, LE_{lt}, and

LE_{sc} for monotonic changes, linear trend and a step change, respectively.

Table 10 shows that the chi-square statistic is above the upper control limit at the sampling time of 24. Also, the likelihood estimations depict that the best estimation belongs to the designed estimator for monotonic changes with $\widehat{\tau} = 9$ while the maximum value of estimations for estimator proposed by Nedumaran and Pignatiello [6] matches with i=18 and the estimator derived for drift approximates the time of change at i=3 (see the bold values in Table 10). Figure 8 depicts the χ^2control chart and the accuracy of estimators based on Table 10.

5 Conclusion

In this paper, we proposed a maximum likelihood estimator by assuming that some kind of monotonic changes are introduced to the mean vector of a multivariate normal process. Also to make a collation between the performances of our proposed estimator with two other estimators which are designed for step changes and linear trend shifts, series of Mont Carlo simulations were implemented to compare the accuracy and precision of the estimations. Numerical examples showed that if a step change occurs in the mean vector the MLE proposed by Nedumaran and Pignatiello [6] outperforms the other ones. However, for small magnitudes of change our proposed estimator showed a better performance than the MLE designed for linear trend shifts. Also, if the mean vector of process increases linearly then the MLE derived for linear trend changes is the one with the best accuracy. Although, for small values of slope of change our proposed estimator outperforms the MLE designed for step change. When three step changes are imposed to the mean vector, our suggested estimator relatively outperforms the other estimators especially for small magnitudes of change at the first change point. Finally, it is worthy to recall that knowing the type of change is a principal assumption in deriving the MLE of step change and linear trend shifts while the exact type of change is not a necessity in deriving the MLE for monotonic changes and this can be a strength point of our proposed MLE in application of such estimators.

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