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On nearly Kenmotsu manifolds

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Abstract: We prove that on a nearly Kenmotsu manifold a second-order symmetric closed recurrent tensor is a multiple of the associated metric tensor. We then find the necessary condition under which a vector field on a nearly Kenmotsu manifold will be a strict contact or Killing vector field. Finally, we prove that every ϕ-recurrent nearly Kenmotsu manifold is an Einstein manifold and every locally ϕ-recurrent nearly Kenmotsu manifold is a manifold of constant curvature -1.

Key words: Almost contact metric, Kenmotsu manifold, nearly Kenmotsu manifold

1. Introduction

An almost contact metric manifold $(M^{2m+1}, \phi, \xi, \eta, g)$ is called a nearly Kenmotsu manifold by Shukla [9] if the following relation holds:

$$(\nabla_X \phi)Y + (\nabla_Y \phi)X = -\eta(Y)\phi X - \eta(X)\phi Y, \tag{1.1}$$

where ∇ is the Levi–Civita connection of g. Moreover, if M satisfies

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{1.2}$$

then it is called a Kenmotsu manifold [4]. It is easy to see that every Kenmotsu manifold is a nearly Kenmotsu manifold but the converse is not true. A nearly Kenmotsu manifold is not a K-contact manifold and hence is not a Sasakian manifold [8]. Recently, nearly Kenmotsu manifolds have been studied extensively; see [1], [7], and [11].

Tensor algebras play a prominent role in differential geometry, in particular in Riemannian geometry. Wong studied recurrent tensor fields on a manifold endowed with a linear connection [12]. Levy showed that on a space of constant curvature, second-order symmetric parallel nonsingular tensors are constant multiples of the metric tensor [6]. Here, we prove that on a nearly Kenmotsu manifold, every second-order closed recurrent tensor whose recurrence covector annihilates ξ is a multiple of the metric tensor (see Theorem 3.1).

It is well known that geometric vector fields on Riemannian manifolds reveal many aspects of those manifolds. In particular, (strict) contact vector fields on an almost contact manifold represent the symmetries of that structure on the underlying manifold [3]. Here, we prove that on a nearly Kenmotsu manifold, every contact vector field on a nearly Kenmotsu manifold leaving the Ricci tensor invariant is a strict contact vector

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field (see Theorem 3.2) and every vector field leaving the curvature tensor invariant is a Killing vector field (see Theorem 3.3).

Finally, we show that every ϕ-recurrent nearly Kenmotsu manifold is an Einstein manifold (see Theorem 4.2) and every locally ϕ-recurrent nearly Kenmotsu manifold is of constant curvature -1 (see Theorem 4.3).

2. Preliminaries

First, we recall some important identities holding in every n-dimensional nearly Kenmotsu manifold (M, ϕ, η, ξ, g) (for more details, see [7]):

$$\phi\xi = 0, \ \ \eta(\xi) = 1, \ \ \phi^2 X = -X + \eta(X)\xi, \ \ \eta \circ \phi = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \ \ g(X, \xi) = \eta(X), \tag{2.2}$$

$$g(X, \phi Y) = -g(\phi X, Y), \ \ (\nabla_X \eta)Y = g(X, Y) - \eta(X)\eta(Y), \tag{2.3}$$

$$R(\xi, X)Y = -g(X, Y)\xi + \eta(Y)X, \ \ R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.4}$$

$$S(\phi X, \phi Y) = S(X, Y) - (1 - n)\eta(X)\eta(Y), \tag{2.5}$$

$$\eta(R(X, Y)Z) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X), \tag{2.6}$$

where R is the Riemannian curvature and S is the Ricci tensor of g.

Let (M, ϕ, η, ξ, g) be a nearly Kenmotsu manifold. In [5], it is proven that the following relations hold:

$$\nabla_X \xi = X - \eta(X)\xi, \ \ \nabla_\xi \xi = 0. \tag{2.7}$$

Hence, the integral curves of ξ are the geodesics of g.

A $(0, 2)$-tensor field α on a Riemannian manifold (M, g) is said to be a recurrent tensor if α satisfies $\nabla\alpha = \lambda \otimes \alpha$ for some 1-form λ. The 1-form λ is called the recurrence covector of α. It is easy to see that every multiple of the metric tensor is a recurrent tensor. Moreover, if λ is a closed 1-form, then α is called a closed recurrent tensor. The set of closed recurrent tensors contains the set of parallel tensors ($\lambda = 0$) as a subset (for more details, see 2 often cited papers, [12][13]).

A vector field X on a nearly Kenmotsu manifold (M, ϕ, ξ, η, g) is said to be

- a contact vector field, if
$$\mathcal{L}_X \eta(Y) = \sigma\eta(Y), \tag{2.8}$$
- or a conformal vector field, if
$$\mathcal{L}_X g(Y, Z) = \rho g(Y, Z), \tag{2.9}$$

where σ and ρ are scalar functions on M and $\mathcal{L}_X$ denotes the Lie derivative along X. X is then called a strict contact vector field or a Killing vector field if $\sigma = 0$ or $\rho = 0$, respectively [3].

A nearly Kenmotsu manifold is said to be a locally ϕ-symmetric manifold in the sense of Takahashi [10] if for all vector fields X, Y, Z, W orthogonal to ξ the following holds:

$$\phi^2((\nabla_W R)(X, Y)Z) = 0, \tag{2.10}$$

and it is said to be a ϕ-recurrent manifold (locally ϕ-recurrent manifold, resp.) if there exists a nonzero 1-form B such that

$$\phi^2((\nabla_W R)(X, Y)Z) = B(W)R(X, Y)Z \tag{2.11}$$

for arbitrary vector fields X, Y, Z, W (for all X, Y, Z, W orthogonal to ξ, resp.).

3. Tensor fields on nearly Kenmotsu manifolds

In this section, we study recurrent tensor fields of the second order and some kind of geometric vector fields on a nearly Kenmotsu manifold.

3.1. Recurrent tensor fields of the second order

Suppose that (M,ϕ,η,ξ,g) is a nearly Kenmotsu manifold and α is a closed recurrent $(0,2)$-tensor on M whose recurrence covector annihilates ξ, i.e. $\lambda(\xi)=0$. Then a straightforward computation yields the following:

$$\alpha(R(W,X)Y,Z)+\alpha(Y,R(W,X)Z)=\lambda(W)\alpha(\nabla_X Y,Z)-\lambda(X)\alpha(\nabla_W Y,Z), \tag{3.1}$$

for arbitrary vector fields X,Y,Z,W on M. Taking $Y=Z=W=\xi$ in (3.1) and using (2.7), we get

$$\alpha(R(\xi,X)\xi,\xi)+\alpha(\xi,R(\xi,X)\xi)=0, \tag{3.2}$$

in which we have used $\lambda(\xi)=0$. Using (2.4) in (3.2), we have

$$2g(X,\xi)\alpha(\xi,\xi)-\alpha(X,\xi)-\alpha(\xi,X)=0. \tag{3.3}$$

Differentiating (3.3) along Y and using $\nabla_\xi\xi=0$ yields

$$\begin{aligned}2\{g(\nabla_Y X,\xi)+g(X,\nabla_Y\xi)\}\alpha(\xi,\xi)&=\alpha(\nabla_Y X,\xi)+\alpha(X,\nabla_Y\xi)\\&\quad+\alpha(\nabla_Y\xi,X)+\alpha(\xi,\nabla_Y X).\end{aligned} \tag{3.4}$$

Replacing X by $\nabla_Y X$ in (3.3), we find

$$2g(\nabla_Y X,\nabla_Y\xi)\alpha(\xi,\xi)-\alpha(\nabla_Y X,\xi)-\alpha(\xi,\nabla_Y X)=0. \tag{3.5}$$

From (3.4) and (3.5), we obtain

$$2g(X,\nabla_Y\xi)\alpha(\xi,\xi)=\alpha(X,\nabla_Y\xi)+\alpha(\nabla_Y\xi,X). \tag{3.6}$$

Using (2.7) implies that

$$2g(X,Y-\eta(Y)\xi)\alpha(\xi,\xi)=\alpha(X,Y-\eta(Y)\xi)+\alpha(Y-\eta(Y)\xi,X). \tag{3.7}$$

From (3.3) and (3.7), one can obtain

$$\alpha^s(X,Y)=\alpha(\xi,\xi)g(X,Y), \tag{3.8}$$

where α^s denotes the symmetric part of α, i.e.

$$\alpha^s(X,Y)=\frac{1}{2}\{\alpha(X,Y)+\alpha(Y,X)\}.$$

On the other hand, using (3.3) and $\nabla\alpha=\lambda\otimes\alpha$, we get $\nabla_X\mu=\lambda(X)\mu$, where X is an arbitrary vector field on M and $\mu=\alpha(\xi,\xi)$. Thus, if α is a parallel tensor or equivalently $\lambda=0$, then μ is a constant function. However, in general μ is not a constant function. Moreover, if α is symmetric, i.e. $\alpha=\alpha^s$, then we get $\alpha=\mu g$, and $\lambda=d\mu$. Hence, we have the following.

Theorem 3.1 *On a nearly Kenmotsu manifold $(M^{2m+1},\phi,\xi,\eta,g)$, a second-order symmetric closed recurrent tensor whose recurrence covector annihilates ξ is a multiple of the metric tensor g.*

3.2. Geometric vector fields on nearly Kenmotsu manifolds

Suppose that a contact vector field X leaves the Ricci tensor invariant, i.e.

$$\mathcal{L}_X S(Y,Z) = 0. \tag{3.9}$$

It follows from (3.9) that

$$\mathcal{L}_X (S(Y,\xi)) = S(\mathcal{L}_X Y,\xi) + S(Y,\mathcal{L}_X \xi). \tag{3.10}$$

Using (2.5), (2.8), and (3.10) implies that

$$(1-n)\sigma\eta(Y) = S(Y,\mathcal{L}_X \xi)). \tag{3.11}$$

Putting $Y=\xi$ in (3.11) and using (2.5), we get

$$\sigma = \eta(\mathcal{L}_X \xi). \tag{3.12}$$

On the other hand, substituting ξ for Y in (2.8) yields

$$\sigma = -\eta(\mathcal{L}_X \xi). \tag{3.13}$$

Hence, $\sigma = 0$.

Theorem 3.2 *Every contact vector field on a nearly Kenmotsu manifold leaving the Ricci tensor invariant is a strict contact vector field.*

Now, let X be a vector field on a nearly Kenmotsu manifold (M,ϕ,η,ξ,g) and $\mathcal{L}_X R = 0$. It is known that the curvature tensor of g is antisymmetric in its 2 last arguments with respect to g, i.e.

$$g(R(U,V)Y,Z) + g(R(U,V)Z,Y) = 0. \tag{3.14}$$

Applying $\mathcal{L}_X$ to (3.14), one can obtain

$$\mathcal{L}_X(g(R(U,V)Y,Z)) + \mathcal{L}_X(g(R(U,V)Z,Y)) = 0. \tag{3.15}$$

Putting $U=Y=Z=\xi$ in (3.15) and using (2.4), we get

$$\mathcal{L}_X g(V,\xi) = \eta(V)\mathcal{L}_X g(\xi,\xi). \tag{3.16}$$

Again putting $U=Y=\xi$ in (3.15) and using (2.4), we get

$$\mathcal{L}_X g(V,Z) - \eta(V)\mathcal{L}_X g(\xi,Z) + \eta(Z)\mathcal{L}_X g(V,\xi) - \mathcal{L}_X g(\xi,\xi)g(V,Z) = 0. \tag{3.17}$$

It follows from (3.16) and (3.17) that

$$\mathcal{L}_X g = \rho g, \tag{3.18}$$

where $\rho = \mathcal{L}_X g(\xi,\xi)$. From (2.5) and the assumption $\mathcal{L}_X R = 0$, which implies that $\mathcal{L}_X S = 0$, we get

$$\rho = \mathcal{L}_X g(\xi,\xi) = -2g(\mathcal{L}_X \xi,\xi) = \frac{2}{n-1} S(\mathcal{L}_X \xi,\xi) = \frac{1}{1-n}\mathcal{L}_X S(\xi,\xi) = 0. \tag{3.19}$$

Theorem 3.3 *Every vector field on a nearly Kenmotsu manifold leaving the curvature tensor invariant is a Killing vector field.*

4. ϕ-Recurrent nearly Kenmotsu manifolds

Let us consider a ϕ-recurrent nearly Kenmotsu manifold $(M^{2m+1}, \phi, \eta, \xi, g)$ $(m > 1)$. Then by virtue of (2.4) and (2.11), we have

$$(\nabla_W R)(X,Y)Z = \eta((\nabla_W R)(X,Y)Z)\xi - B(W)R(X,Y)Z. \tag{4.1}$$

From (4.1) and the Bianchi identity, we get

$$B(W)\eta(R(X,Y)Z) + B(X)\eta(R(Y,W)Z) + B(Y)\eta(R(W,X)Z) = 0. \tag{4.2}$$

Let $\{e_i\}$ be an orthonormal basis of the tangent space at any point of the manifold M. Putting $Y = Z = e_i$ in (4.2) and taking summation over i, we get by virtue of (2.4)

$$B(W)\eta(X) = B(X)\eta(W), \tag{4.3}$$

for all vector fields X, W. Replacing X by ξ in (4.3), it follows that

$$B(W) = \eta(\hat{B})\eta(W), \tag{4.4}$$

since $B(\xi) = g(\xi, \hat{B}) = \eta(\hat{B})$. Now, suppose that M is η-Einstein, i.e.

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{4.5}$$

where a and b are 2 scalar functions on M. Putting $Y = \xi$ in both (2.3) and (4.5) yields

$$a + b = 1 - n. \tag{4.6}$$

In a local coordinate, (4.5) can be written as follows:

$$R_{ij} = ag_{ij} + b\eta_i\eta_j, \tag{4.7}$$

which implies that

$$r = (2m+1)a + b. \tag{4.8}$$

Taking the covariant derivative with respect to g from (4.7) implies that

$$R_{ij,k} = a_{,k}g_{ij} + b_{,k}\eta_i\eta_j + b\eta_{i,k}\eta_j + b\eta_i\eta_{j,k}. \tag{4.9}$$

Contracting (4.9) with g^{ik}, we get

$$R^k_{j,k} = a_{,j} + b_{,k}\xi^k\eta_j + b\eta_{i,k}g^{ik}\eta_j + b\eta_i\eta_{j,k}g^{ik}. \tag{4.10}$$

It is well known that $R^k_{j,k} = \frac{1}{2}r_{,j}$. Hence, we get

$$r_{,j} = 2\{a_{,j} + (b_{,k}\xi^k + 2mb)\eta_j\}, \tag{4.11}$$

in which we have used (2.3ii) and $\eta_{i,k}g^{ik} = (g_{ik} - \eta_i\eta_k)g^{ik} = 2m$. On the other hand, taking the covariant derivative of (4.6) and (4.8) yields

$$r_{,j} = 2ma_{,j}. \tag{4.12}$$

Plugging (4.12) into (4.11), we get

$$ma_{,j} = a_{,j} + (b_{,k}\xi^k + 2mb)\eta_j. \tag{4.13}$$

Contracting (4.13) with ξ^j and using (4.6), we get

$$b_{,k}\xi^k = -2b. \tag{4.14}$$

Moreover, if b or a is a constant function, then (4.14) implies that $b = 0$. Thus, M is an Einstein manifold. More precisely, we have the following.

Proposition 4.1 *Let $(M^n, \phi, \xi, \eta, g)$ be an η-Einstein nearly Kenmotsu manifold $(n = 2m + 1, m > 1)$. Suppose that b or a is a constant function. Then M is an Einstein manifold.*

From (4.1), we have

$$-g(\nabla_W R)(X, Y)Z, U) + \eta((\nabla_W R)(X, Y)Z)\eta(U) = B(W)g(R(X, Y)Z, U). \tag{4.15}$$

Let $\{e_i\}$ be an orthonormal basis for the tangent space of M at a point $p \in M$. Putting $X = U = e_i$ in (4.15) and taking summation over i, we get

$$-(\nabla_W S)(Y, Z) + A_W(Y, Z) = B(W)S(Y, Z), \tag{4.16}$$

where $A_W(Y, Z) = \sum_{i=1}^{n} \eta((\nabla_W R)(e_i, Y)Z)\eta(e_i)$. We claim that $A_W(Y, \xi) = 0$. First, we recall that $\eta((\nabla_W R)(e_i, Y)\xi) = g((\nabla_W R)(e_i, Y)\xi, \xi)$. Now, we have

$$\begin{aligned} g((\nabla_W R)(e_i, Y)\xi, \xi) = {} & g(\nabla_W R(e_i, Y)\xi, \xi) - g(R(\nabla_W e_i, Y)\xi, \xi) \\ & - g(R(e_i, \nabla_W Y)\xi, \xi) - g(R(e_i, Y)\nabla_W \xi, \xi). \end{aligned} \tag{4.17}$$

Evaluating (4.17) at $p \in M$ and using $g_{ij}(p) = \delta_{ij}$, we get $\nabla_W e_i(p) = 0$. We also have

$$g(R(e_i, \nabla_W Y)\xi, \xi) = -g(R(\xi, \xi)\nabla_W Y, e_i) = 0, \tag{4.18}$$

since R is skew-symmetric. Using (4.18) and $\nabla_W e_i(p) = 0$ in (4.17), we obtain

$$g((\nabla_W R)(e_i, Y)\xi, \xi) = g(\nabla_W R(e_i, Y)\xi, \xi) - g(R(e_i, Y)\nabla_W \xi, \xi). \tag{4.19}$$

By virtue of $g(R(e_i, Y)\xi, \xi) = -g(R(\xi, \xi)Y, e_i) = 0$, we have

$$g(\nabla_W R(e_i, Y)\xi, \xi) + g(R(e_i, Y)\xi, \nabla_W \xi) = 0, \tag{4.20}$$

which implies

$$g((\nabla_W R)(e_i, Y)\xi, \xi) = -\{g(R(e_i, Y)\xi, \nabla_W \xi) + g(R(e_i, Y)\nabla_W \xi, \xi)\} = 0, \tag{4.21}$$

since R is skew-symmetric. This means that $\eta((\nabla_W R)(e_i, Y)\xi) = 0$ and consequently $A_W(Y, \xi) = 0$, and from (4.16)

$$(\nabla_W S)(Y, \xi) = -B(W)S(Y, \xi). \tag{4.22}$$

On the other hand, by definition, we have

$$(\nabla_W S)(Y, \xi) = \nabla_W S(Y, \xi) - S(\nabla_W Y, \xi) - S(Y, \nabla_W \xi). \tag{4.23}$$

Applying (2.3ii), (2.5), and (2.7) in (4.23), we get

$$(\nabla_W S)(Y,\xi) = -(n-1)g(Y,W) - S(Y,W). \tag{4.24}$$

Substituting (4.24) into (4.22) and using (4.4), we obtain

$$S(Y,W) = ag(Y,W) + b\eta(Y)\eta(W), \tag{4.25}$$

where $a = 1-n$ and $b = (1-n)\eta(\hat{B})$. This means that the manifold is an η-Einstein manifold with $a = constant$. By Proposition 4.1, it follows that M is an Einstein manifold. Therefore, we have the following.

Theorem 4.2 *Every ϕ-recurrent nearly Kenmotsu manifold is an Einstein manifold.*

It is known that on a nearly Kenmotsu manifold we have the following [2]:

$$(\nabla_W R)(X,Y)\xi = g(W,X)Y - g(W,Y)X - R(X,Y)W. \tag{4.26}$$

By virtue of (2.6), it follows from (4.26) that

$$\eta((\nabla_W R)(X,Y)\xi) = 0. \tag{4.27}$$

In view of (4.26) and (4.27), we obtain from (4.1)

$$-(\nabla_W R)(X,Y)\xi = B(W)R(X,Y)\xi, \tag{4.28}$$

from which by using (4.26), it follows that

$$-g(X,W)Y + g(Y,W)X + R(X,Y)W = B(W)R(X,Y)\xi. \tag{4.29}$$

Hence, if X and Y are orthogonal to ξ, then we get from (2.4ii)

$$R(X,Y)\xi = 0. \tag{4.30}$$

Thus, we obtain

$$R(X,Y)W = -\{g(Y,W)X - g(X,W)Y\}, \tag{4.31}$$

for all X, Y, W. Thus, we have the following.

Theorem 4.3 *A locally ϕ-recurrent nearly Kenmotsu manifold $(M^{2m+1}, \phi, \xi, \eta, g)$ $(m > 1)$ is a manifold of constant curvature -1.*

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