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# Ricci almost soliton on Conformally flat pr-waves manifolds

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**Abstract**

A generalization of the Ricci solitons was recently considered. The purpose of this paper is to investigate the existence of non-trivial Ricci almost solitons on conformally flat pr-waves manifolds.

**Keywords:** Ricci almost solitons, pr-waves manifolds, Conformally flat

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## 1 Introduction

The study of an Ricci almost soliton was introduced in a recent paper due to Pigola et al.[4], where essentially they modified the definition of Ricci solitons by adding the condition on the parameter $\lambda$ to be a variable function. More precisely,

**Definition 1.1.** A Riemannian manifold $(M^n, g)$ is a Ricci almost soliton if there exist a vector field $X$ and a soliton function $\lambda : M^n \to \mathbb{R}$ satisfying

$$Ric + \frac{1}{2} L_X g = \lambda g \tag{1}$$

where $Ric$ and $L$ stand, respectively, for the Ricci tensor and the Lie derivative.

Ricci almost soliton will be called expanding, steady or shrinking, respectively, if $\lambda < 0, \lambda = 0 or \lambda > 0$. Otherwise it will be called indefinite. When the vector field $X$ is a gradient of a differentiable function $f : M^n \to \mathbb{R}$ the manifold will be called a gradient Ricci almost soliton; in this case the preceding equation turns out to be

$$Ric + \nabla^2 f = \lambda g \tag{2}$$

where $\nabla^2 f$ stands for the Hessian of $f$. Sometimes the theory of the classical tensorial calculus is more convenient to make computations. Then we shall write the fundamental equation in this language as follows:

$$Ric_{ij} + \nabla_i \nabla_j f = \lambda g_{ij}. \tag{3}$$

[1]speaker

Moreover, when either the vector field X is trivial or the potential $f$ is constant, the Ricci almost soliton will be called trivial, while for a nontrivial Ricci almost soliton its potential function is not trivial. We notice that when $n \geq 3$ and $X$ is a Killing vector field, an Ricci almost soliton will be simply a Ricci soliton, since in this case we have an Einstein manifold, which implies that $\lambda$ is a constant. Taking into account that the soliton function $\lambda$ is not necessarily constant, certainly the comparison with the soliton theory will be modified. In fact, we refer the reader to [4] to see some of these changes.

A Lorentzian manifold with a parallel light-like vector field is called Brinkmann-wave, due to [2]. A Brinkmann-wave manifold $(M, g)$ is called pp-wave if its curvature tensor $R$ satisfies the trace condition $tr_{(3,5)(4,6)}(R \otimes R) = 0$. In [3], Schimming proved that an $(n+2)-$dimensional pp-wave manifold admits coordinates $(x, y_1, ..., y_n, z)$ such that g has the form $g = dxdz + \sum_{k=1,...,n}(dy_k)^2 + f(dz)^2$ with $\partial_x f = 0$. In [5], Leistner gave another equivalence for pp-wave manifold. More precisely, he proved that a Brinkmann-wave manifolds $(M, g)$ with parallel light-like vector field $X$ and induced parallel distributions $E$ and $E^{\perp}$ is a pp-wave if and only if its curvature tensor satisfies

$$R(U, V) : E^{\perp} \to E, \tag{4}$$

for all $U, V \in TM$. or equivalently $R(Y_1, Y_2) = 0$ for all $Y_1, Y_2 \in E^{\perp}$. From this description, it follows that a pp-wave manifold is Ricci-isotropic, which means that the image of the Ricci operator is totally light-like, and has vanishing scalar curvature [5]. Furthermore, Leistner introduced a new class of non-irreducible Lorentzian manifolds satisfying (4) but only for a recurrent vector field $X$, that is,$\nabla X = \omega \bigotimes .X$ where $\omega$ is a one-form on $M$. Following [5], such manifolds are called pr-waves. Moreover, a description in terms of local coordinates similar to the one for pp-waves manifolds was given in [5]: a Lorentzian manifold $(M, g)$ of dimension $n+2 > 2$ is a pr-wave if and only if around any point $o \in M$ exist coordinates $(x, y_1, ..., y_n, z)$ in which the metric g has the following form:

$$g = dxdz + \sum_{k=1,...,n} (dy_k)^2 + f(dz)^2, \tag{5}$$

where $f$ is a real valued smooth function on $(M, g)$.

## 2 Main results

In this section we want to find a solution for Ricci almost solution on the a class of pr-waves manifolds $(M, g)$.
We denote by $\nabla$ the Levi Civita connection of $(M, g)$ and by $R$ its curvature tensor, taken with the sign convention:

$$R(X, Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]. \tag{6}$$

As it is well known, given a pseudo-Riemannian manifold $(M, g)$, its Ricci tensor $Ric$ is defined as the contraction of the curvature tensor:

$$Ric(X, Y) = trZ \to R(X, Z)Y. \tag{7}$$

Throughout the paper we shall assume $n \geq 1$. With respect to the basis of coordinate vector fields $\{\partial_x, \partial_{y_1}, ..., \partial_{y_n}, \partial_z\}$ for which (5) holds, the non-vanishing components of the metric are

$$g_{y_i y_i} = 1 \quad for \;\; 1 \leq i \leq n \quad g_{xz} = 1 \quad g_{zz} = f. \tag{8}$$

Put $f_x := \partial_x f, f_{y_k} := \partial_{y_k} f$ and $f_z := \partial_z f$. By (8), a standard calculation gives that the only possible non-vanishing Christoffel symbols are the following ones:

$$\Gamma^x_{xz} = -\Gamma^z_{zz} = \frac{f_x}{2}, \Gamma^x_{zz} = \frac{1}{2}(ff_x + f_z), \Gamma^x_{y_i z} = -\Gamma^{y_i}_{zz} = \frac{f_{y_i}}{2}, \quad i = 1, ..., n. \tag{9}$$

Next, using (9) in (6) and (7) we obtain that the non-vanishing components of the Ricci tensor, with respect to $\{\partial_x, \partial_{y_1}, ..., \partial_{y_n}, \partial_z\}$, are the following ones:

$$\begin{aligned} Ric_{xz} &= Ric(\partial_x, \partial_z) = \frac{1}{2} f_{xx}, \\ Ric_{y_i z} &= Ric(\partial_x, \partial_{y_i}) = \frac{1}{2} f_{xy_i}, \qquad for 1 \leq i \leq n \\ Ric_{zz} &= Ric(\partial_z, \partial_z) = \frac{1}{2}(ff_{xx} - \sum_{k=1,...,n} f_{y_k y_k}). \end{aligned} \tag{10}$$

In this paper we consider the case of conformally flat pr-waves manifolds $(M, g)$, that is the defining function is given by

$$f = K(z)x + \sum_{k=1,...,n} (G(z)\frac{y_k}{2} + F_k(z))y_k + H(z), \tag{11}$$

where $K, G, F_k$ and $H$ are smooth functions on $(M, g)$ depending on $z$[1]. Let $X = A\partial_x + \sum_{k=1,...,n} B_k \partial_{y_k} + C\partial_z$ be an arbitrary vector field on $M$, where $A, B_k, C$ are smooth functions of the variables $x, y_1, ..., y_n, z$. Then, the Lie derivative of the metric (5) is given by:

$$\begin{aligned} (L_X g)(\partial_x, \partial_x) &= 2\partial_x C, \\ (L_X g)(\partial_x, \partial_{y_i}) &= \partial_x B_i + \partial_{y_i} C, \\ (L_X g)(\partial_x, \partial_z) &= \partial_x A + f\partial_x C + \partial_z C, \\ (L_X g)(\partial_{y_i}, \partial_{y_j}) &= \partial_{y_i} B_j + \partial_{y_j} B_i, \\ (L_X g)(\partial_{y_i}, \partial_z) &= \partial_{y_i} A + f\partial_{y_i} C + \partial_z B_i, \\ (L_X g)(\partial_z, \partial_z) &= Af_x + \sum_{k=1,...,n} B_k f_{y_k} + Cf_z + 2(\partial_z A + f\partial_z C). \end{aligned} \tag{12}$$

**Theorem 2.1.** *Any non-flat locally conformally flat pr-wave admits suitable vector fields resulting in expanding, steady, and shrinking Ricci almost solitons.*

*Proof.* By using (8), (7) and (12) in (1) and taking into account (11), a standard calculation gives that a conformally flat pr-wave manifold $(M, g)$ is Ricci almost soliton if and only if the following system holds

$$\begin{aligned} \partial_x C &= 0, \\ \partial_x B_i + \partial_{y_i} C &= 0, \\ \partial_x A + f\partial_x C + \partial_z C &= \lambda, \\ \partial_{y_i} B_j + \partial_{y_j} B_i &= 2\lambda\delta^j_i \\ \partial_{y_i} A + f\partial_{y_i} C + \partial_z B_i &= 0 \\ Af_x + \sum_{k=1,...,n} B_k f_{y_k} + Cf_z + 2(\partial_z A + f\partial_z C) &= 2\lambda f. \end{aligned} \tag{13}$$

where $f$ is given by (11) and $i, j = 1, ..., n$. The first equation in (13) yields $C = C(y_1, ..., y_n, z)$. Deriving the second equation in (13) with respect to $y_j$, we have

$$\partial_x \partial_{y_j} B_i + \partial_{y_i} \partial_{y_j} C = 0 for all i, j = 1, ..., n.$$

From the fourth equation in (13) and taking into account that $\partial_x \partial_{y_j} B_i = -\partial_{y_i} \partial_{y_j} C$, we get

$$\partial_x \partial_{y_i} B_j = \partial_{y_i} \partial_{y_j} C = 0 \qquad for \quad all \quad i, j = 1, ..., n. \tag{14}$$

Next, we derive the fifth equation in (13) with respect to $y_j$ and using (14), we obtain

$$\partial_{y_i} \partial_{y_j} A + f_{y_j} \partial_{y_i} C + \partial_{y_j} \partial_z B_i = 0. \tag{15}$$

Thus we have also

$$\partial_{y_j} \partial_{y_i} A + f_{y_i} \partial_{y_j} C + \partial_{y_j} \partial_z B_j = 0. \tag{16}$$

By (15), (16) and using the fourth equation in (13), we have

$$f_{y_j} \partial_{y_i} C - f_{y_i} \partial_{y_j} C + 2 \partial_{y_j} \partial_z B_j = 0.$$

which, by derivation with respect to $y_i$, gives (since $f_{y_i y_j} = 0$ for $i \neq j$ and $\partial_{y_i} B_i = \lambda$)

$$f_{y_i y_i} \partial_{y_j} C + 2 \partial_{y_j} \partial_z \lambda = 0 \qquad for \quad all \quad i \neq j. \tag{17}$$

Now, by derivation with respect to $x$ from $\partial_{y_i} B_i = \lambda$ and using (14) we get $\lambda = \lambda(y_1, ..., y_n, z)$. We derive the fourth equation in (13) with respect to $y_k$, we get

$$\partial_{y_k} \partial_{y_i} B_j + \partial_{y_k} \partial_{y_j} B_i = 0 \qquad for \quad all \quad i, j, k = 1, ..., n,$$

and so,

$$\begin{aligned} \partial_{y_i} \partial_{y_k} B_j + \partial_{y_i} \partial_{y_j} B_k &= 0, \\ \partial_{y_j} \partial_{y_i} B_k + \partial_{y_j} \partial_{y_k} B_i &= 0, \end{aligned}$$

which gives $\partial_{y_j} \partial_{y_k} B_i = 0$ for all $i, j, k$.

Assume now $\frac{\partial \lambda}{\partial z} = 0$ therefore, $\lambda$ reduces to a function of the variables $y_1, y_2, ..., y_n, \lambda = \lambda(y_1, ..., y_n)$. It makes $\partial_z \partial_{y_i} B_i = \frac{\partial \lambda}{\partial z} = 0$ and also using (17), considering that $f_{y_i y_i} \neq 0$ we obtain $C = C(z)$. Taking into a account the fact that $\partial_x B_i = \partial_{y_j} \partial_z B_i = 0$, we prove that

$$B_i = \lambda(y_1, ..., y_n) + \sum_{k \neq i} b_{ik} y_k + \tilde{B}_i(z) \quad for all i = 1, ..., n, \tag{18}$$

where $(b_{ik})$ is an arbitrary skew-symmetric matrix and $\tilde{B}_1, ..., \tilde{B}_n$ are smooth functions on $M$.
The third and the fifth equations in (13) imply, since $\partial_{y_i} C = \partial_{y_i} \partial_{y_j} A = 0$,

$$A = (2\lambda(y_1, ..., y_n) - \acute{C}(z))x - \sum_{k=1,...,n} \acute{\tilde{B}}_k(z) y_k + \tilde{A}(z), \tilde{A} \in C^\infty(M) \tag{19}$$

Next, by (18),(19) and using (11), a straightforward calculation gives that the last equation in (13) is satisfied if and only if two systems of equations for function $C, \tilde{B}_i, \tilde{A}, K, G$ holds. Consider the vector field

$$X = [\lambda(y_1, ..., y_n)x - \sum_{k=1,...,n} \tilde{\dot{B}}_k(z)y_k + \tilde{A}(z)]\partial_x + \sum_{k=1,...,n} [\frac{\lambda}{2}y_k + \tilde{B}_k(z)]\partial_{y_k},$$

where functions $\tilde{B}_i$ and $\tilde{A}$ satisfy the following conditions

$$\begin{aligned} \tilde{B}_i G - K\tilde{\dot{B}}_i - 2\tilde{B}''_i &= \frac{\lambda}{2}F_i, \quad for \;\; all \quad i = 1, ..., n, \\ \tilde{A}K + 2\tilde{\dot{A}} + \sum_{k=1,...,n} \tilde{\dot{B}}_k F_k &= \lambda H + \frac{n}{2}G. \end{aligned} \tag{20}$$

From the local existence and uniqueness theorem for ordinary differential equations of first and second orders, it follows that there exists a unique smooth solution $\tilde{B}_i$ to the i-th equation in (20) with given initial conditions and replacing all these solutions into the last equation one also concludes the existence of a smooth solution $\tilde{A}(z)$ to such equation. Thus, we proved that any conformally flat pr-wave manifold admits appropriate vector fields for which (1) holds. □

**Remark 2.2.** Simple calculation shows that the condition $\lambda = \lambda(z, y_1, ..., y_n)$ at the above theorem can not reduce to $\lambda = \lambda(z)$.

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