New bounds on the independence number of connected graphs

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The independence number of a graph $G$, denoted by $\alpha(G)$, is the maximum cardinality of an independent set of vertices in $G$. [Henning and Löwenstein An improved lower bound on the independence number of a graph, Discrete Applied Mathematics 179 (2014) 120–128.] proved that if a connected graph $G$ of order $n$ and size $m$ does not belong to a specific family of graphs, then $\alpha(G) > \frac{5}{8}n - \frac{1}{4}m$. In this paper, we strengthen the above bound for connected graphs with maximum degree at least three that have a non-cut-vertex of maximum degree. We show that if a connected graph $G$ of order $n$ and size $m$ has a non-cut-vertex of maximum degree then $\alpha(G) \geq \frac{5}{8}n - \frac{1}{4}(m - \Delta(G)) - \frac{11}{12}$, where $\Delta(G)$ is the maximum degree of the vertices of $G$. We also characterize all connected graphs $G$ of order $n$ and size $m$ that have a non-cut-vertex of maximum degree and $\alpha(G) \leq \frac{5}{8}n - \frac{1}{4}(m - \Delta(G)) - \frac{2}{3}$.

Keywords: Transversal; independence; maximum degree.

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1. Introduction

In this paper, we study the independence number and the transversal number in graphs. Let $G = (V, E)$ be a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. We denote by $n(G)$ and $m(G)$, or just $n$ and $m$ if $G$ is specified, the order and size of $G$, respectively. For a vertex $v \in V(G)$, let $N_G(v) = \{u \mid uv \in E(G)\}$ denote the open neighborhood of $v$ and $N_G[v] = \{v\} \cup N_G(v)$ denote the closed neighborhood of $v$. The degree of a vertex $v$, $\deg_G(v)$, or just $\deg(v)$, in a graph $G$ is the number of neighbors of $v$ in $G$. We refer to $\Delta(G)$ and $\delta(G)$ as the maximum degree and the minimum degree of the vertices of $G$, respectively. A vertex $v$ in a

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connected graph $G$ is called a cut-vertex if $G - v$ is disconnected. A 2-connected graph is a connected graph with no cut-vertex.

A set $S$ of vertices in a graph $G$ is an independent set if no pair of vertices of $S$ are adjacent. The independence number of $G$, denoted by $\alpha(G)$, is the maximum cardinality of an independent set in $G$. An independent set of cardinality $\alpha(G)$ is called an $\alpha(G)$-set. A vertex covers an edge if it is incident with the edge. A transversal in $G$ is a set of vertices that covers all the edges of $G$. We remark that a transversal is also called a vertex-cover in the literature. The transversal number of $G$, denoted by $\tau(G)$, is the minimum cardinality of a transversal in $G$. A transversal of cardinality $\tau(G)$ is called a $\tau(G)$-set.

We denote by $C_n$, $P_n$ and $K_n$ the cycle, the path, and the complete graph of $n$ vertices, respectively.

2. Known Results

The following observation is well known.

**Observation 1.** For any graph $G$ of order $n$, $\alpha(G) + \tau(G) = n$.

The independence number is one of the most fundamental and well-studied graph parameters (see, for example, [2–16, 18–23, 25, 26]). Many researchers considered graphs with special conditions. Griggs [10] improved the well-known Caro-Wei bound [4, 26] for triangle-free graphs not including odd paths or cycles. Jones [20] considered triangle-free graphs with maximum degree at most four. Heckman and Thomas [17] provided a short proof for the result conjectured by Albertson, Bollobás and Tucker [1] and originally proved by Staton [24] that every triangle-free graph $G$ of maximum degree at most three has an independent set of cardinality at least $5\alpha(G)/14$. Harant et al. [13] extended the result of Heckman and Thomas about the independence number of triangle-free graphs of maximum degree at most three to the case of graphs which may contain triangles.

Recently, Henning and Löwenstein [18] showed that if $G$ does not belong to a specific family of graphs, $\mathcal{G}$, then $\alpha(G) > \frac{2}{3}n - \frac{1}{4}m$. Further, they characterized all graphs $G$ for which $\alpha(G) \leq \frac{2}{3}n - \frac{1}{4}m$. For these purposes, Henning and Löwenstein [18] defined a family $\mathcal{G}$ of graphs. They denoted by $K_4^*$ the graph obtained from $K_4$ by subdividing one edge of the graph twice. In this paper, we call each of the vertices of degree two of $K_4^*$ a divider vertex.

For $t \in \{2, 3, 4\}$ they defined a $K_t$-unit to be a graph isomorphic to $K_t$. Further, a graph isomorphic to $C_5$ and $K_4^*$ they called a $C_5$-unit and a $K_4^*$-unit, respectively. A unit is a $F$-unit for some graph $F \in \{K_2, K_3, K_4, C_5, K_4^*\}$.

Let $\mathcal{G}_{2,1}$ be the family of connected graphs $G$ that can be obtained from $k \geq 1$ disjoint $K_3$-units by adding $k-1$ edges. They noted that each added edge is a bridge of $G$. Let $\mathcal{G}_3 = \mathcal{G}_{3,1}$.

For $k \geq 1$, let $\mathcal{G}_{2}$ be the family of connected graphs that can be obtained from a $K_4$-unit and $k-1$ disjoint $K_3$-units by adding $k-1$ edges. Let $\mathcal{G}_2 = \mathcal{G}_{2,1}$.
For \( k \geq 2 \), let \( \mathcal{G}_{1,1} \) be the family of connected graphs that can be obtained from two disjoint \( K_4 \)-units and \( k - 2 \) disjoint \( K_3 \)-units by adding \( k - 1 \) edges. For \( k \geq 1 \), let \( \mathcal{G}_{1,2} \) be the family of connected graphs that can be obtained from a \( K_2 \)-unit and \( k - 1 \) disjoint \( K_3 \)-units by adding \( k - 1 \) edges. For \( k \geq 1 \), let \( \mathcal{G}_{1,3} \) be the family of connected graphs that can be obtained from a \( C_5 \)-unit and \( k - 1 \) disjoint \( K_3 \)-units by adding \( k - 1 \) edges. Let

\[
\mathcal{G}_1 = \bigcup_{i=1}^{3} \mathcal{G}_{1,i}.
\]

For \( k \geq 3 \), let \( \mathcal{G}_{0,1} \) be the family of connected graphs that can be obtained from three disjoint \( K_4 \)-units and \( k - 3 \) disjoint \( K_3 \)-units by adding \( k - 1 \) edges. For \( k \geq 2 \), let \( \mathcal{G}_{0,2} \) be the family of connected graphs that can be obtained from the disjoint union of a \( K_2 \)-unit, a \( K_4 \)-unit and \( k - 2 \) disjoint \( K_3 \)-units by adding \( k - 1 \) edges. For \( k \geq 2 \), let \( \mathcal{G}_{0,3} \) be the family of connected graphs that can be obtained from the disjoint union of a \( K_2 \)-unit, a \( K_4 \)-unit and \( k - 2 \) disjoint \( K_3 \)-units by adding \( k - 1 \) edges. For \( k \geq 1 \), let \( \mathcal{G}_{0,4} \) be the family of connected graphs that can be obtained from the disjoint union of a \( K_3 \)-unit and \( k - 1 \) disjoint \( K_3 \)-units by adding \( k - 1 \) edges. For \( k \geq 3 \), let \( \mathcal{G}_{0,5} \) be the family of connected graphs that can be obtained from \( k \) disjoint \( K_3 \)-units by adding \( k \) edges. Let

\[
\mathcal{G}_0 = \bigcup_{i=1}^{5} \mathcal{G}_{0,i}.
\]

Finally, they defined the infinite family \( \mathcal{G} \) of graphs by

\[
\mathcal{G} = \bigcup_{i=0}^{3} \mathcal{G}_i.
\]

A graph in the family \( \mathcal{G} \) is illustrated in Fig. 1.

**Theorem 2 (Henning and Löwenstein, [18]).** Let \( G \) be a connected graph of order \( n \) and size \( m \). Then the following holds.

1. If \( G \notin \mathcal{G} \), then \( \tau(G) < \frac{1}{3}n + \frac{1}{3}m \) and \( \alpha(G) > \frac{2}{3}n - \frac{1}{3}m \).
2. If \( G \in \mathcal{G}_i \) where \( i \in \{0, 1, 2, 3\} \), then \( \tau(G) = \frac{1}{3}n + \frac{1}{3}m + \frac{1}{12} \) and \( \alpha(G) = \frac{2}{3}n - \frac{1}{3}m - \frac{1}{12} \).

![Fig. 1. A graph in the family \( \mathcal{G}_1 \).](image-url)
Our aim in this paper is to present new bounds for the independence number of a graph that has a non-cut-vertex with maximum degree. Note that infinite families of graphs have the non-cut-vertex condition, such as 2-connected graphs.

Our results improve Theorem 2 for graphs with maximum degree at least 3. We note that the study of the independence number for graphs with maximum degree at least 3 is of sufficient interest, since for graphs with maximum degree at most two (paths and cycles) the independence number is clear.

3. Families of Graphs

We first present some definitions and a new infinite family of graphs based on the family $\mathcal{G}$. For a graph $G \in \mathcal{G}$, a bridge-alternating path is a path that alternates between edges in units and bridges in $G$. The units that have an edge in a bridge-alternating path $P$ are called to be saturated by $P$, or just $P$-saturated units. A bridge-alternating path whose end edges are not bridge is an bridge-augmenting path. For a bridge-alternating path $P = v_0, v_1, \ldots, v_l$, we denote by $\text{Unit}_s(P)$ the set of vertices in the $P$-saturated units. Denote $N_s(P) = \text{Unit}_s(P) \cup \{v_1, \ldots, v_{l-1}\}$.

We now define an infinite family of graphs based on the family $\mathcal{G}$ as follows.

For $i \in \{0, 1, 2, 3\}$, $j \in \{1, 2, 5\}$ with $(i, j) \notin \{(1, 5), (2, 2), (2, 5), (3, 2), (3, 5)\}$, let $\mathcal{H}_{i,j}$ be the family of connected graphs $G$ that $G$ is obtained from a graph $G_0 \in \mathcal{G}_{i,j}$ by adding a new vertex $x$ and joining $x$ to some vertices of $G_0$ such that $x$ has maximum degree in $G$, and there is an integer $k_G \geq 1$ such that the following procedure holds.

**Procedure $\mathcal{A}$**.

Step 1: There exists a bridge-augmenting path $P^1(x_0) : x_0 = v^1_0, \ldots, v^1_{2i+1} \in G_0$ such that if $k_G = 1$ then $N_s(P^1(x_0)) \subseteq N_G(x)$, and if $k_G > 1$ then $I_1(x_0) \neq \emptyset$, where $I_1(x_0) = N_s(P^1(x_0)) \setminus N_G(x)$ and the following Step 2 holds.

Step 2: For any vertex $x_1 \in I_1(x_0)$, there exists a bridge-alternating path $P^2(x_1) : x_1 = v^2_1, \ldots, v^2_{2l+1} \in G_0$ such that $v^2_1v^2_2$ is a bridge of $G_0$, if $k_G = 2$ then $N_s(P^2(x_1)) \subseteq N_G(x)$ for all $x_1 \in I_1(x_0)$, and if $k_G > 2$ then $I_2(x_1) \neq \emptyset$ for some $x_1 \in I_1(x_0)$, where $I_2(x_1) = N_s(P^2(x_1)) \setminus N_G(x)$ and the following Step 3 holds for all $x_1 \in I_1(x_0)$ such that $I_2(x_1) \neq \emptyset$.

Step 3: For any vertex $x_{k-1} \in I_{k-1}(x_{k-2})$, there exists a bridge-alternating path $P^k(x_{k-1}) : x_{k-1} = v^k_1, \ldots, v^k_{2l+1} \in G_0$ such that $v^k_1v^k_2$ is a bridge of $G_0$, if $k_G = k$ then $N_s(P^k(x_{k-1})) \subseteq N_G(x)$ for all $x_{k-1} \in I_{k-1}(x_{k-2})$, and if $k_G > k$ then $I_k(x_{k-1}) \neq \emptyset$ for some $x_{k-1} \in I_{k-1}(x_{k-2})$, where $I_k(x_{k-1}) = N_s(P^k(x_{k-1})) \setminus N_G(x)$, and the Step $k+1$ holds for all $x_{k-1} \in I_{k-1}(x_{k-2})$ such that $I_k(x_{k-1}) \neq \emptyset$.

We note that the final step of the Procedure $\mathcal{A}$ is as follows: For any vertex $x_{k_G-1} \in I_{k_G-1}(x_{k_G-2})$, there exists a bridge-alternating path $P^{k_G}(x_{k_G-1}) : x_{k_G-1} = v^{k_G}_1, \ldots, v^{k_G}_{2l+1} \in G_0$ such that $v^{k_G}_1v^{k_G}_2$ is a bridge of $G_0$, and if $k_G > k$ then $I_{k_G}(x_{k_G-1}) \neq \emptyset$ for some $x_{k_G-1} \in I_{k_G-1}(x_{k_G-2})$, where $I_{k_G}(x_{k_G-1}) = N_s(P^{k_G}(x_{k_G-1})) \setminus N_G(x)$, and the Step $k_G+1$ holds for all $x_{k_G-1} \in I_{k_G-1}(x_{k_G-2})$ such that $I_{k_G}(x_{k_G-1}) \neq \emptyset$. 
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Fig. 2. Some graphs in the family \( \mathcal{H}_3 \) obtained by Procedure \( \mathcal{A} \), letting \( G_0 \) be the graph shown in Fig. 1.

\[ x_{kG-1} = v_1^{kG}, \ldots, v_2^{kG+1} \] in \( G_0 \) such that \( v_1^{kG}, v_2^{kG} \) is a bridge of \( G_0 \), and \( N_i(P^{kG}(x_{kG-1})) \subseteq N_G(x) \) for all \( x_{kG-1} \in I_{kG}(x_{kG-2}) \).

For \( i \in \{0, 1\} \), \( j \in \{3, 4\} \) with \( (i, j) \neq (1, 4) \), let \( \mathcal{H}_{i,j} \) be the family of connected graphs \( G \) that \( G \) is obtained from a graph \( G_0 \in \mathcal{G}_{i,j} \) by adding a new vertex \( x \) and joining \( x \) to some vertices of \( G_0 \) such that \( x \) has maximum degree in \( G \), and there is an integer \( k_G \geq 1 \) such that at least one of the following procedures hold.

**Procedure B.** The Steps 1, \ldots, \( k_G \) of the Procedure \( \mathcal{A} \) hold subject to there is no \( P^{kG}(x-1) \)-saturated \( F \)-unit for \( F \in \{C_5, K_4^*\} \), in any Step 1 \( \leq k \leq k_G \).

**Procedure C.** There is a vertex \( x_0 \) of the \( F \)-unit, where \( F \in \{C_5, K_4^*\} \), such that if \( k_G = 1 \) then \( N_G[x_0] \cap V(F) \subseteq N_G(x) \), and if \( k_G > 1 \) then \( I_1(x_0) \neq \emptyset \), where \( I_1(x_0) = (N_G[x_0] \cap V(F)) \setminus N_G(x) \) and the Step 2, \ldots, \( k_G \) of the Procedure \( \mathcal{A} \) hold.

We now define an infinite family \( \mathcal{H} \) of graphs by

\[ \mathcal{H} = \bigcup_{i=0}^{3} \mathcal{H}_i, \]

where \( \mathcal{H}_3 = \mathcal{H}_{3,1}, \mathcal{H}_2 = \mathcal{H}_{2,1}, \mathcal{H}_1 = \bigcup_{j=1}^{3} \mathcal{H}_{1,j} \), and \( \mathcal{H}_0 = \bigcup_{j=1}^{5} \mathcal{H}_{0,j} \). Some graphs in the family \( \mathcal{H} \) are illustrated in Fig. 2. As mentioned before, graphs in the family \( \mathcal{H} \) are constructed using graphs in the family \( \mathcal{G} \), but they have different structures.

4. Main Results

By the construction of the graphs in the family \( \mathcal{G} \), the following has an straightforward proof.

**Observation 3.** If \( G \in \{C_5, K_4^*\} \) then there is no \( \tau(G) \)-set containing the closed neighborhood of a vertex of \( G \).

**Observation 4.** If \( G \in \mathcal{G} \setminus \mathcal{G}_{0,5} \) then there is no cycle in \( G \) containing a bridge of \( G \).

The following can be obtained from Theorem 2. (See Appendix for proof).

**Corollary 5.** If \( G \in \mathcal{G} \), then every \( \tau(G) \)-set has exactly \( \tau(F) \) vertices from every \( F \)-unit in \( G \).
Proposition 6. Let $G$ be a connected graph of order $n$ and size $m$ that has a non-cut-vertex of maximum degree. Then

$$
\tau(G) \leq \frac{1}{3}n + \frac{1}{4}(m - \Delta(G)) + \frac{11}{12}.
$$

Proof. Let $G$ be a connected graph of order $n$ and size $m$ that has a non-cut-vertex $x$ of maximum degree. Let $G_0 = G - x$. Then $G_0$ is a connected graph of order $n - 1$ and size $m - \Delta(G)$. By Theorem 2, we have

$$
\tau(G) \leq 1 + \tau(G_0)
$$

$$
\leq 1 + \frac{1}{3}(n - 1) + \frac{1}{4}(m - \Delta(G)) + \frac{1}{4}
$$

$$
= \frac{1}{3}n + \frac{1}{4}(m - \Delta(G)) + \frac{11}{12}.
$$

We shall prove the following result, a proof of which is given in Sec. 5.

Theorem 7. Let $G$ be a connected graph of order $n$ and size $m$ that has a non-cut-vertex of maximum degree. Then the following hold.

(a) $\tau(G) = \frac{1}{3}n + \frac{1}{4}(m - \Delta(G)) + \frac{1}{4}(8 + i)$ (and thus $\alpha(G) = \frac{2}{9}n - \frac{1}{9}(m - \Delta(G)) - \frac{1}{3}(8 + i)$, where $i \in \{0, 1, 2, 3\}$, if and only if $G \in \mathcal{H}_i$.

(b) If $G \not\in \mathcal{H}_i$, then $\tau(G) < \frac{1}{3}n + \frac{1}{4}(m - \Delta(G)) + \frac{2}{3}$ (and thus $\alpha(G) > \frac{2}{9}n - \frac{1}{9}(m - \Delta(G)) - \frac{2}{3}$).

Note that a simple calculation shows that Theorem 7 improves Theorem 2 if $\Delta(G) \geq 3$. Also, clearly Theorem 7 holds for any 2-connected graph.

5. Proof of Theorem 7

We now prove Theorem 7. It is sufficient to prove (a).

$(\Leftarrow)$ Let $G \in \mathcal{H}_i$, where $i \in \{0, 1, 2, 3\}$, be a graph of order $n$ and size $m$. Thus $G$ is obtained from a graph $G_0 \in \mathcal{G}_i$ by adding a new vertex $x$ and one of the Procedures $A$, $B$ and $C$. We show that no $\tau(G_0)$-set covers $E(G)$. On the contrary, let $S$ be a $\tau(G_0)$-set that covers $E(G)$. We proceed with two cases according to the value of $j$ which $G \in \mathcal{H}_{i,j}$.

Case I. $G \in \mathcal{H}_{i,j}$ for some $j \in \{1, 2, 5\}$, and $(i, j) \not\in \{(1, 5), (2, 2), (2, 5), (3, 2), (3, 5)\}$. Then $G$ is obtained by Procedure $A$ and each unit in $G_0$ is a $K_t$-unit, where $t \in \{2, 3, 4\}$.

Claim 1. $I_t(x_0) \subseteq S$.

Proof. By Step $k_G$, for any $x_{k_{t-1}} \in I_{k_{t-1}}(x_{k_{t-2}})$, there is a bridge-alternating path $P^G_{k_G}(x_{k_{t-1}}) : x_{k_{t-1}} = v_1^{k_G}, \ldots, v_{2t_{k_{t-1}}+1}^{k_G}$, such that $v_{2l}^{k_G} \in \mathcal{V}_{2l+1}^{k_G}$, where $1 \leq l \leq k_G$.
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is an edge of a $F_t^{kG}$-unit, and $N_s(P_t^{kG}(x_{kG-1})) \subseteq N_G(x)$, for all $x_{kG-1} \in I_{kG-1}(x_{kG-2})$. Since $S$ covers $E(G)$ and $x \notin S$, we deduce that $N_G(x) \subseteq S$. Then

$$N_s(P_t^{kG}(x_{kG-1})) \subseteq S. \quad (1)$$

Therefore, we have $N_G(v_{2kG-1}^{kG}) \cap N_s(P_t^{kG}(x_{kG-1})) \subseteq S$. Since the $F_t^{kG}$-unit is a $K_t$-unit, where $t \in \{2, 3, 4\}$, we have

$$|N_G(v_{2kG-1}^{kG}) \cap N_s(P_t^{kG}(x_{kG-1}))| = |V(F_t^{kG}) \setminus \{v_{2kG-1}^{kG}\}| = n(F_t^{kG}) - 1.$$

Therefore, by Corollary 5, $v_{2kG-1}^{kG} \notin S$. Since $S$ covers the bridge $v_{2kG-1}^{kG} v_{2kG-2}^{kG}$, we find that $v_{2kG-1}^{kG} \notin S$. By (1), $N_G(v_{2kG-1}^{kG}) \cap N_s(P_t^{kG}(x_{kG-1})) \subseteq S$. Then we obtain that

$$\left(N_G(v_{2kG-1}^{kG}) \cap N_s(P_t^{kG}(x_{kG-1})) \right) \cup \{v_{2kG-1}^{kG}\} \subseteq S.$$ 

Since the $F_t^{kG}$-unit is a $K_t$-unit, where $t \in \{2, 3, 4\}$, we have

$$|\left(N_G(v_{2kG-1}^{kG}) \cap N_s(P_t^{kG}(x_{kG-1})) \right) \cup \{v_{2kG-1}^{kG}\}| = |V(F_t^{kG}) \setminus \{v_{2kG-1}^{kG}\}|$$

$$= n(F_t^{kG}) - 1.$$

Therefore, by Corollary 5, $v_{2kG-2}^{kG} \notin S$. Since $S$ covers the bridge $v_{2kG-3}^{kG} v_{2kG-2}^{kG}$, we find that $v_{2kG-2}^{kG} \notin S$. Proceeding this argument for $N_G(v_{2kG+1}^{kG}) \cap N_s(P_t^{kG}(x_{kG-1}))$, $1 \leq l \leq l_{kG} - 2$, we obtain that $v_{2kG+1}^{kG} \notin S$. Hence, $x_{kG-1} = v_{kG}^{kG} \in S$. Then

$$I_{kG-1}(x_{kG-2}) \subseteq S. \quad (2)$$

For any $x_{kG-2} \in I_{kG-2}(x_{kG-3})$, there is a bridge-alternating path $P_{kG-1}(x_{kG-2})$ : $x_{kG-2} = v_{kG-1}^{kG}, \ldots, v_{2kG+1}^{kG}$, such that $v_{2l+1}^{kG} v_{2l+2}^{kG}$, where $1 \leq l \leq l_{kG-1}$, is an edge of a $F_t^{kG-1}$-unit. (2) together with the fact that $N_G(x) \subseteq S$, implies that $N_s(P_{tG-1}(x_{kG-2})) \subseteq S$. Then

$$N_G(v_{2(kG-1)+1}^{kG}) \cap N_s(P_{tG-1}(x_{kG-2})) \subseteq S.$$

Since the $F_t^{kG-1}$-unit is a $K_t$-unit, where $t \in \{2, 3, 4\}$, we have

$$|N_G(v_{2(kG-1)+1}^{kG}) \cap N_s(P_{tG-1}(x_{kG-2}))| = |V(F_t^{kG-1}) \setminus \{v_{2(kG-1)+1}^{kG}\}|$$

$$= n(F_t^{kG-1}) - 1.$$

Therefore, by Corollary 5, $v_{2(kG-1)+1}^{kG} \notin S$. Since $S$ covers the bridge $v_{2(kG-1)+1}^{kG} v_{2(kG-1)+2}^{kG}$, we have $v_{2(kG-1)+1}^{kG} v_{2(kG-1)+2}^{kG} \notin S$. Proceeding this argument for $N_G(v_{2(kG+1)}^{kG}) \cap N_s(P_{tG-1}(x_{kG-2}))$, $1 \leq l \leq l_{kG-1} - 1$, we obtain that $v_{2l-1}^{kG} \in S$. Hence, $x_{kG-2} = v_{kG-1}^{kG} \in S$. Then

$$I_{kG-2}(x_{kG-3}) \subseteq S. \quad (3)$$

An analogous argument similar those applied to prove (1), (2) and (3) yields that $I_{kG}(x_{kG-(k+1)}) \subseteq S$ for $1 \leq k \leq kG - 1$, as desired.
There is a bridge-augmenting path \( P^1(x_0) : x_0 = v_0^1, \ldots, v_{2l+1}^1 \), where \( v_{2l+1}^1 \) is an edge of a \( F^1 \)-unit. Claim 1 together with the fact that \( N_G(x) \subseteq S \), implies that \( N_s(P^1(x_0)) \subseteq S \). Since \( F_0^1 \)-unit is a \( K_1 \)-unit, where \( t \in \{2, 3, 4\}, V(F_0^1) = N_G(v_0^1) \). So by definition of \( N_s(P^1(x_0)) \), we have

\[
V(F_0^1) \setminus \{v_1^1\} = N_G[v_0^1] \cap N_s(P^1(x_0)) \subseteq S. 
\]  

(4)

Similar to the proof of Claim 1, we can see that for \( 1 \leq l \leq l_1, v_{2l+1}^1 \in S \). Then \( v_1^1 \in S \). This together with (4) implies that \( V(F_0^1) \subseteq S \). This is a contradiction with Corollary 5.

**Case II.** \( G \in H_{i,j} \), where \( j \in \{3, 4\} \) and \( (i,j) \neq (1,4) \). By Case I, we may assume that \( G \) is obtained by Procedure \( C \). The Steps 2, \ldots, \( k_2 \) of the Procedure \( C \) are identical to the steps in the Procedure \( A \). Then by Claim 1, we have \( I_1(x_0) \subseteq S \). Therefore, for a vertex \( x_0 \) in a \( F \)-unit, where \( F \in \{C_5, K_4^*\} \), we have \( N_G[x_0] \cap V(F) \subseteq S \). This is a contradiction with Observation 3.

By Case I, \( G \not\in H_{i,j} \), for \( j \in \{1, 2, 5\} \) and \( (i,j) \notin \{(1,5), (2,2), (2,5), (3,2), (3,5)\} \), and by Case II, \( G \not\in H_{i,j} \), for \( j \in \{3, 4\} \) and \( (i,j) \neq (1,4) \). Hence, \( G \not\in H \), a contradiction. Thus there is no \( \tau(G_0) \)-set that covers \( E(G) \). We deduce that \( \tau(G) = 1 + \tau(G_0) \). By Theorem 2, we have

\[
\tau(G) = 1 + \tau(G_0) \\
= 1 + \frac{1}{3}(n-1) + \frac{1}{4}(m - \Delta(G)) + \frac{i}{12} \\
= \frac{1}{3}n + \frac{1}{4}(m - \Delta(G)) + \frac{8+i}{12}. 
\]

Consequently, by Observation 1, \( \alpha(G) = \frac{2}{3}n - \frac{1}{4}(m - \Delta(G)) - \frac{1}{2}(8+i) \).

(\(\Rightarrow\)) Let \( G \) be a connected graph of order \( n \) and size \( m \) that has a non-cut-vertex \( x \) of maximum degree and \( \tau(G) = \frac{2}{3}n + \frac{1}{4}(m - \Delta(G)) + \frac{2}{3}(8+i) \), where \( i \in \{0, 1, 2, 3\} \). We show that \( G \in H \). Let \( G_0 = G - x \). Then \( G_0 \) is a connected graph of order \( n-1 \) and size \( m - \Delta(G) \). Suppose that \( G_0 \not\in \mathcal{G} \). Then by Theorem 2,

\[
\tau(G_0) \leq 1 + \tau(G_0) < 1 + \frac{1}{3}(n-1) + \frac{1}{4}(m - \Delta(G)) \\
= \frac{1}{3}n + \frac{1}{4}(m - \Delta(G)) + \frac{2}{3}, 
\]

a contradiction, since \( \tau(G) \geq \frac{2}{3}n + \frac{1}{4}(m - \Delta(G)) + \frac{2}{3} \). Hence, \( G_0 \in \mathcal{G} \). Therefore, there is an integer \( j \in \{0, 1, 2, 3\} \) such that \( G_0 \in \mathcal{G}_j \).

**Claim 2.** There is no \( \tau(G_0) \)-set that covers \( E(G) \).

**Proof.** If there exists a \( \tau(G_0) \)-set that covers \( E(G) \), then \( \tau(G) = \tau(G_0) \). By Theorem 2, we have

\[
\tau(G) = \tau(G_0) = \frac{1}{3}(n-1) + \frac{1}{4}(m - \Delta(G)) + \frac{j}{12} = \frac{1}{3}n + \frac{1}{4}(m - \Delta(G)) + \frac{j-4}{12}. 
\]

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This implies that \( j - i = 12 \), which is impossible, since \( j - i \leq 3 \). Therefore, no \( \tau(G_0) \)-set covers \( E(G) \).

By Claim 2 and Theorem 2,

\[
\tau(G) = 1 + \tau(G_0) = \frac{1}{3}n + \frac{1}{4}(m - \Delta(G)) + \frac{8 + j}{12}.
\]

Therefore, \( i = j \), i.e., \( G_0 \in \mathcal{G} \).

We show that there is an integer \( k \) such that \( G \) is obtained from \( G_0 \) by Steps 1, 2, \ldots, \( k \) of one of the Procedures \( \mathcal{A}, \mathcal{B} \) and \( \mathcal{C} \). Let \( S \) be a \( \tau(G_0) \)-set that has maximum number of vertices of \( N_G(x) \) in the graph \( G \). By Claim 2, \( S \) does not cover \( E(G) \). So \( N_G(x) \setminus S \neq \emptyset \). Let \( v_0 \in N_G(x) \setminus S \), and let \( v_0 \) be a vertex of a \( F_1 \)-unit. Since the \( F_1 \)-unit is a connected graph of order at least two, then \( N_G(v_0) \cap V(F_1) \neq \emptyset \). Also,

\[
N_G(v_0) \cap V(F_1) \subseteq S,
\]

(5) since \( S \) covers \( E(G_0) \) and \( v_0 \notin S \). For each \( v \in N_G(v_0) \cap V(F_1), \) let \( S^v = (S \setminus \{v\}) \cup \{v_0\} \).

**Claim 3.** For every vertex \( v \in N_G(v_0) \cap V(F_1), \) \( N_G(v) \setminus S^v \neq \emptyset \).

**Proof.** Clearly, \(|S^v| = |S| = \tau(G_0)\), for any vertex \( v \in N_G(v_0) \cap V(F_1) \). Suppose that there is a vertex \( v \in N_G(v_0) \cap V(F_1) \) such that \( N_G(v) \setminus S^v = \emptyset \). Hence, \( v \notin N_G(x) \) and \( S^v \) covers \( E(G_0) \). Then \( S^v \) is a \( \tau(G_0) \)-set that has more vertices of \( N_G(x) \) than \( S \), a contradiction to the choice of \( S \). Therefore, for every vertex \( v \in N_G(v_0) \cap V(F_1), \) \( v \) is adjacent to \( x \) or \( S^v \) does not cover \( E(G_0) \). In other words, every vertex \( v \in N_G(v_0) \cap V(F_1) \) is adjacent to \( x \) or is adjacent to a vertex in \( V(G_0) \setminus S^v \).

Set \( I_{v_0} = (N_G(v_0) \cap V(F_1)) \setminus N_G(x) \). Let \( I_{v_0} = \emptyset \). Hence,

\[
N_G(v_0) \cap V(F_1) \subseteq N_G(x).
\]

(6) If \( F_0 \)-unit is a \( K_t \)-unit, where \( t \in \{2, 3, 4\} \), then the edge \( v_0v \), where \( v \in N_G(v_0) \cap V(F_1) \), is a bridge-augmenting path \( P^1(v_0) \) with length 1. Also, by (6),

\[
N_4(P^1(v_0)) = V(F_0) = N_G(v_0) \cap V(F_1) \subseteq N_G(x).
\]

Then \( G \) is obtained by Step 1 of Procedure \( \mathcal{A} \) or Procedure \( \mathcal{B} \). If \( F_0 \in \{C_5, K_4^*\} \),

then by (6), \( G \) is obtained by Step 1 of Procedure \( \mathcal{C} \). Therefore, \( G \in \mathcal{H}_t \) and the proof of the theorem is complete. Next assume that \( I_{v_0} \neq \emptyset \). We now consider two cases according to the value of \( j \) which \( G_0 \in \mathcal{G}_{i,j} \).

**Case 1.** \( G_0 \in \mathcal{G}_{i,j} \), where \( j \in \{1, 2, 5\} \) and \( (i, j) \notin \{(1, 5), (2, 2), (2, 5), (3, 2), (3, 5)\} \).

Then the units in \( G_0 \) are \( K_t \)-units, where \( t \in \{2, 3, 4\} \). By Claim 3, for every vertex \( v \in I_{v_0}, \) there is a vertex \( w \in N_G(v) \setminus S^v \).
Claim 4. For every vertex \( v \in I_{v_0^1} \) and \( w \in N_{G_0}(v) \setminus S_{v_0^1} \), the edge \( vw \) is a bridge of \( G_0 \).

**Proof.** On the contrary, suppose that there is a vertex \( v \in I_{v_0^1} \) and a vertex \( w \in N_{G_0}(v) \setminus S_{v_0^1} \) such that \( w \in V(F_{i_0}^1) \). Since the \( F_{i_0}^1 \)-unit is a \( K_t \)-unit, where \( t \in \{ 2, 3, 4 \} \), we have \( N_G(v_0^1) \cap V(F_{i_0}^1) = V(F_{i_0}^1) \setminus \{ v_0^1 \} \). Hence, by (5) \( V(F_{i_0}^1) \setminus \{ v_0^1 \} \subseteq S \). So by the definition of \( S_{v_0^1} \), \( V(F_{i_0}^1) \setminus \{ v_0^1 \} \subseteq S_{v_0^1} \). Then \( w \in S_{v_0^1} \), a contradiction. Therefore, for any \( v \in I_{v_0^1} \) and \( w \in N_{G_0}(v) \setminus S_{v_0^1} \), \( v \not\in V(F_{i_0}^1) \), and so \( vw \) is a bridge of \( G_0 \). \( \Box \)

For every vertex \( v \in I_{v_0^1} \), we have \( S \setminus \{ v \} \subseteq S_{v_0^1} \). Also \( S \) covers \( E(G_0) \). Hence, \( S_{v_0^1} \) covers \( E(G_0) \). Therefore, for every vertex \( w \in N_{G_0}(v) \setminus S_{v_0^1} \) in a \( F \)-unit, we have \( N_G(w) \cap V(F) \subseteq S_{v_0^1} \). Since each unit is a connected graph of order at least two, we have \( N_G(w) \cap V(F) \neq \emptyset \). For every vertex \( w \in N_{G_0}(v) \setminus S_{v_0^1} \), choose a vertex \( v_0^1 \in N_G(w) \cap V(F) \), then set \( A = \{ v_0^1 | w \in N_{G_0}(v_0^1) \setminus S_{v_0^1} \} \). This contradicts the choice of \( S \). Thus, there is a vertex \( w \in N_{G_0}(v) \setminus S_{v_0^1} \) in a \( F \)-unit such that for every vertex \( w \in N_G(v) \cap V(F) \) is adjacent to \( x \) or is adjacent to a vertex in \( V(G_0) \setminus S_{v_0^1} \). Therefore, for every \( w \in N_{G_0}(v) \setminus S_{v_0^1} \), we have \( S_{v_0^1} \setminus \{ v_0^1 \} \subseteq S_{v_0^1} \) and \( z \in N_G(w) \cap V(F) \), \( N_G(z) \setminus S_{v_0^1} \neq \emptyset \). \( \Box \)

Let \( v_1^1 \) be a vertex of \( I_{v_0^1} \). By Claim 5, there is a vertex \( v_2^1 \in N_G(v_1^1) \setminus S_{v_0^1} \) in a \( F_j \)-unit such that for every vertex \( v \in N_G(v_2^1) \cap V(F_{i_0}^1) \), \( N_G(v) \setminus S_{v_0^1} \neq \emptyset \).

Set \( I_{v_1^1} = (N_G(v_1^1) \cap V(F_{i_0}^1)) \setminus N_G(x) \). Assume that \( I_{v_1^1} = \emptyset \). Since the \( F_j \)-unit is a connected graph of order at least two, we have \( N_G(v_2^1) \cap V(F_{i_0}^1) \neq \emptyset \). By Claim 4, the path \( P(v_0^1) : v_0^1, v_1^1, v_2^1, v \), where \( v \in N_G(v_2^1) \cap V(F_{i_0}^1) \), is a bridge-augmenting path. Set \( I_1(v_0^1) = (I_{v_0^1} \cup I_{v_1^1}) \setminus \{ v_0^1 \} \). Clearly, \( I_1(v_0^1) = N_G(P(v_0^1)) \setminus N_G(x) \). If \( I_1(v_0^1) = \emptyset \), then \( G \in \mathcal{H}_{i,j} \), where \( j \in \{ 1, 2, 5 \}, (i,j) \not\in \{ (1,5), (2,2), (2,5), (3,2), (3,5) \} \), and \( G \) is obtained by Step 1 of Procedure \( A \). Therefore, \( G \in \mathcal{H}_i \) and the proof of the theorem is complete. If \( I_1(v_0^1) \neq \emptyset \), then for every vertex \( v_3^1 \in I_1(v_0^1) \), \( v_3^1 \in
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By Claim 4, \( v_7^2 \) is adjacent to a vertex \( w \in V(G_0) \setminus S_{v_3}^{v_1} \).

By Claim 4, \( v_7^2w \) is a bridge of \( G_0 \). By Claim 5, there is a vertex \( v_2^2 \in N_G(v_7^2) \setminus S_{v_3}^{v_1} \)
in a \( F_2^1 \)-unit such that for every vertex \( v \in N_G(v_7^2) \cap V(F_2^1), N_G(v) \setminus S_{v_3}^{v_1} \neq \emptyset \).

Set \( I_{v_3} = (N_G(v_7^2) \cap V(F_2^1)) \setminus N_G(x) \). This argument will continue with discussion on the \( I_{v_3} \).
If \( I_{v_3} \neq \emptyset \), since \( G \) is a finite graph, there is an integer \( l_2 \geq 1 \) that \( I_{v_3} = \emptyset \).
Then the path \( P^2(v_7^1) : v_7^1, \ldots, v_{2l_2+1}^2 \) is a bridge-alternating path in \( G_0 \).

Set \( I_2(v_7^1) = \cup_{l=1}^{l_2} I_{v_3} \setminus \{v_2^3, v_2^4, \ldots, v_{2l_2}^2 \} \). Clearly, \( I_2(v_7^1) = N_v(P^2(v_7^1)) \setminus N_G(x) \).

If for a vertex \( v_7^1 \in I_1(v_0^1), I_2(v_7^1) \neq \emptyset \), then continue as procedure of obtaining the path \( P^2(v_7^1) \) with the start of every vertex \( v_7^1 \in I_2(v_7^1) \). Since \( G_0 \) is a finite graph, then there is an integer \( k_0 \) that \( N_v(P^{k_0}(v_7^1)) \subseteq N_G(x) \) for all \( v_7^1 \in I_{k_0} = (v_7^1, k_0) \).

Then, \( G \in H_{i,j} \) where \( j \in \{1, 2, 5\}, (i, j) \notin \{(1, 5), (2, 2), (2, 5), (3, 2), (3, 5)\} \), and \( G \) is obtained by Step 1 of Procedure \( A \). Therefore, \( G \in H_i \) and the proof of the theorem is complete.

Next assume that \( I_{v_3} = \emptyset \). Similar to Claim 3, for every vertex \( v_7^1 \in I_{v_3} \), \( N_G(v_7^1) \setminus S_{v_3}^{v_1} \neq \emptyset \). Since the \( F_1^1 \)-unit is a \( K_1 \)-unit, where \( t \in \{2, 3, 4\} \), similar to Claim 4, for any \( w \in N_G(v_7^1) \setminus S_{v_3}^{v_1} \), the edge \( wv_7^1 \) is a bridge of \( G_0 \). Similar to the proof of Claim 5, there is a vertex \( v_2^3 \in N_G(v_7^1) \setminus S_{v_3}^{v_1} \) in a \( F_1^1 \)-unit such that for every vertex \( v \in N_G(v_7^1) \cap V(F_2^1), N_G(v) \setminus S_{v_3}^{v_1} \neq \emptyset \). Set \( I_{v_3} = (N_G(v_7^1) \cap V(F_2^1)) \setminus N_G(x) \).

This argument will continue with discussion on the \( I_{v_3} \).
If \( I_{v_3} \neq \emptyset \), since \( G_0 \) is a finite graph, there is an integer \( l_1 \geq 2 \) such that for \( 2 \leq l < l_1 \) we have the following.

\( S_{v_7^1}^{v_3} = (S_{v_2}^{v_3} \setminus \{v_{2l_2+1}^2\}) \cup \{v_2^3\} \), where \( v_2^3 \in N_G(v_7^1) \setminus S_{v_3}^{v_1} \) is a vertex of a \( F_2^1 \)-unit and \( v_{2l_2+1}^2 \in N_G(v_2^3) \cap V(F_1^1) \).

Similar to Claims 3 and 4, for \( 2 \leq l < l_1 \), we have

**Claim 6.** For every vertex \( v \in N_G(v_2^1) \cap V(F_1^1), N_G(v) \setminus S_{v_3}^{v_1} \neq \emptyset \).

**Claim 7.** The edge \( v_{2l_2+1}^2w \) is a bridge of \( G_0 \), where \( w \in N_G(v_{2l_2+1}^2) \setminus S_{v_3}^{v_1} \).

For \( 2 \leq l \leq l_1 - 1 \), set \( I_{v_3} = (N_G(v_2^1) \cap V(F_1^1)) \setminus N_G(x) \) such that \( I_{v_3} \neq \emptyset \) and \( I_{v_{2l_2+1}}^2 = \emptyset \).

Since the \( F_1^1 \)-unit is a connected graph of order at least two, then \( N_G(v_2^1) \cap V(F_1^1) \neq \emptyset \). By Claims 4 and 7, the path \( P^1(v_0^1) : v_0^1, v_1^1, \ldots, v_{2l_2}^1 \),
where \( v_{2l_2}^1 \in N_G(v_{2l_2}^1) \cap V(F_1^1) \), is a bridge-augmenting path in \( G_0 \). Set \( I_1(v_0^1) = \cup_{l=0}^{l_2} I_{v_3} \setminus \{v_1^1, v_2^1, \ldots, v_{2l_2}^1\} \).
Clearly, \( I_1(v_0^1) = N_v(P^1(v_0^1)) \setminus N_G(x) \). If \( I_1(v_0^1) = \emptyset \), then \( G \in H_{i,j} \) where \( j \in \{1, 2, 5\}, (i, j) \notin \{(1, 5), (2, 2), (2, 5), (3, 2), (3, 5)\} \), and \( G \) is obtained by Step 1 of Procedure \( A \). Therefore, \( G \in H_i \) and the proof of the theorem is complete. If \( I_1(v_0^1) \neq \emptyset \), for every vertex \( v_7^1 \in I_1(v_0^1) \), there is an integer \( l \in \{0, \ldots, l_1 - 1\} \) that \( v_7^1 \in I_{v_3} \setminus \{v_{2l_2}^1\} \).
By Claim 6, \( v_7^1 \) is adjacent to a vertex \( v_9^2 \in V(G_0) \setminus S_{v_3}^{v_1} \). By Claim 7, \( v_9^2v_2^1 \) is a bridge of \( G_0 \), then \( v_9^2 \) is a vertex of a \( F_2^1 \)-unit. Continue as procedure of obtaining the path \( P^1(v_0^1) \) with the
start of the vertex $v_2^2 \in N_G(v_1^2) \setminus S_{v_1^2}^{k-1}$. Therefore, there is an integer $l_2 \geq 1$ that $I_{v_{2l_2}^2} = \emptyset$. Then the path $P^2(v_1^2) : v_1^2, v_2^2, \ldots, v_{2l_2+1}^2$ is a bridge-alternating path in $G_0$. Set $I_2(v_1^2) = \cup_{i=1}^{2l_2} I_{v_i^2} \setminus \{v_2^2, v_3^2, \ldots, v_{2l_2+1}^2\}$. Clearly, $I_2(v_1^2) = N_G(P^2(v_1^2)) \setminus N_G(x)$. If for each vertex $v_i^2 \in I_1(v_1^2)$, $I_2(v_i^2) = \emptyset$, then $G \in \mathcal{H}_{i,j}$, where $j \in \{1, 2, 5\}$, $(i, j) \notin \{(1, 5), (2, 2), (2, 5), (3, 2), (3, 5)\}$, and $G$ is obtained by Steps 1, 2 of Procedure $A$. Therefore, $G \in \mathcal{H}_i$ and the proof of the theorem is complete. If for a vertex $v_i^2 \in I_1(v_1^2)$, $I_2(v_i^2) \neq \emptyset$, then continue as procedure of obtaining the path $P^2(v_1^2)$ with the start of every vertex $v_i^2 \in I_2(v_1^2)$. Since $G_0$ is a finite graph, then there is an integer $k_0$ that $N_G(v_1^{k_0}) \subset N_G(x)$, for every $v_1^{k_0} \in I_{k-1}(v_1^{k_0})$, then $G \in \mathcal{H}_{i,j}$, where $j \in \{1, 2, 5\}$, $(i, j) \notin \{(1, 5), (2, 2), (2, 5), (3, 2), (3, 5)\}$, and $G$ is obtained by Steps 1, 2, $k_0$ of Procedure $A$. Therefore, $G \in \mathcal{H}_i$ and the proof of the theorem is complete.

**Case 2.** $G_0 \in \mathcal{H}_{i,j}$, where $j \in \{3, 4\}$ and $(i, j) \neq (1, 4)$. We follow the argument used to obtain the paths $P^1, \ldots, P^{k_0}$ in Case 1. Assume that each $F_i^k$-unit is a $K_t$-unit, where $t \in \{2, 3, 4\}$, $1 \leq k \leq k_0$, $0 \leq l \leq l_1$ if $k = 1$, and $1 \leq l \leq l_k$ if $k \geq 2$. Then we proceed to obtain the paths $P^1, \ldots, P^{k_0}$. Consequently, $G$ is obtained by Steps 1, 2, $k_0$ of Procedure $B$. Therefore $G \in \mathcal{H}_{i,j}$, where $j \in \{3, 4\}$ and $(i, j) \neq (1, 4)$, i.e., $G \in \mathcal{H}_i$ and the proof of the theorem is complete. Next assume that there are integers $k \in \{1, \ldots, k_0\}$ and $l \in \{0, \ldots, l_k\}$ such that $v_l^k \in V(F_l^k)$, where $F_l^k \in \{C_5, K_4^*\}$. By the argument used to obtain the paths $P^k$ in Case 1, we have the following fact.

**Fact 1.** $v_l^k \in N_G(x)$ or $v_l^k$ is a vertex of the bridge-alternating path $P : v_1^{k_1}, v_2^{k_1}, \ldots, v_0^k$ (if $k = 1$) that $v_1^{k_1} \cup v_2^{k_1}$ is a bridge of $G_0$ and $v_1^{k_1}$ belongs to the $F_1^k$-unit, or $v_l^k$ is a vertex of the bridge-alternating path $P : v_1^{k'}, v_2^{k'}, \ldots, v_1^{k'}, v_l^{k'}$ (if $k \geq 2$) that $v_l^{k'} \in I_{v_{k-1}^{k'}} \setminus \{v_{k+1}^{k'-1}\}$, $v_l^{k'} \cup v_{l+1}^{k'}$ is a bridge of $G_0$ and $v_l^{k'} \cup v_{l+1}^{k'}$ belongs to the $F_l^{k-1}$-unit.

Similar to the proof of Claim 3, for every vertex $z \in N_G(v_l^k) \cap V(F_l^k)$, $N_G(z) \setminus S_{v_l^k}^{k-1} \neq \emptyset$. Let $\left(N_G(v_l^{k'}) \cap V(F_l^k)\right) \setminus N_G(x) = \emptyset$. Then set $I_l(v_l^{k'}) = (N_G(v_{k-1}^k) \cap V(F_l^k)) \setminus N_G(x)$. Hence, $G$ is obtained by Step 1 of Procedure $C$. Therefore, $G \in \mathcal{H}_{i,j}$, where $j \in \{3, 4\}$ and $(i, j) \neq (1, 4)$, i.e., $G \in \mathcal{H}_i$ and the proof of the theorem is complete. Next we assume that $N_G(v_l^k) \cap V(F_l^k)) \setminus N_G(x) \neq \emptyset$. Let for each vertex $v \in (N_G(v_{k-1}^k) \cap V(F_l^k)) \setminus N_G(x)$, $N_G(v) \setminus S_{v_l^k}^{k-1} \notin V(F_l^k)$. Then for each vertex $v \in (N_G(v_l^k) \cap V(F_l^k)) \setminus N_G(x)$, there is a vertex $w \in N_G(v) \setminus (V(F_l^k) \cup S_{v_l^k}^{k-1})$, so $vw$ is a bridge of $G_0$. Therefore, we can follow the same argument used to obtain the paths $P^2, \ldots, P^{k_0}$ in Case 1, for each $v \in (N_G(v_{k-1}^k) \cap V(F_l^k)) \setminus N_G(x)$, since $j \neq 5$ and so by Observation 4, for the each next unit is a $K_t$-unit, where $t \in \{2, 3, 4\}$. This together with the Fact 1 implies that $G$ is obtained by Steps 1, 2, $k_0$ of Procedure $C$. Therefore, $G \in \mathcal{H}_{i,j}$, where $j \in \{3, 4\}$ and $(i, j) \neq (1, 4)$, i.e., $G \in \mathcal{H}_i$ and the proof of the theorem is complete. Therefore, we assume now that the following holds.

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There is a vertex $v \in (N_G(v^k_i) \cap V(F^k_i)) \setminus N_G(x)$ such that $N_G(v) \setminus S_{c^k_i}^{c^k_i} \subseteq V(F^k_i)$.

We know that $F^k_i \in \{C_5, K^*_4\}$. Let the vertices of $C_5$ and $K^*_4$ are labeled as Fig. 3.

Assume that $F^k_i = C_5$. Without loss of generality, assume that $v^k_i = c_1$ and $v = c_2$. By $(\star)$, $N_G(c_2) \setminus S_{c_2}^{c_2} \neq \emptyset$ and $N_G(c_2) \setminus S_{c_2}^{c_2} \subseteq V(F^k_i)$. Thus assume for the next that $F^k_i = \{c_1, c_3\}$. But $c_1 \in S_{c_2}^{c_2}$, then $c_3 \notin S_{c_2}^{c_2}$.

Subcase 2.1. $v^k_i$ is a divider vertex of $K^*_4$. Now consider the vertex $v$ described in $(\star)$. Assume that $v$ is not a divider vertex. Without loss of generality, we can suppose $v^k_i = c_1$, $v = c_2$. By $(\star)$, $N_G(c_2) \setminus S_{c_2}^{c_2} \neq \emptyset$ and $N_G(c_2) \setminus S_{c_2}^{c_2} \subseteq V(F^k_i)$. $N_G(c_2) \cap V(F^k_i) = \{c_1, c_3, c_4\}$. But $c_1 \in S_{c_2}^{c_2}$, then $c_3, c_4 \notin S_{c_2}^{c_2}$. Hence, $c_3 \notin S_{c_2}^{c_2}$. Therefore, $\{c_3, c_4\} \subseteq S$.

Set $I_1(x) = N_G(c_3) \setminus N_G(x)$. If $I_1(x) = \emptyset$ then $G \in H_1$, and $G$ is obtained by Step 1 of Procedure $C$. Thus, assume that $I_1(x) \neq \emptyset$. By the Fact 1, (7) and (8), we can use the argument of obtaining the paths $P^2, \ldots, P^{k_G}$ as Case 1, since $j = 5$ and so by Observation 4, for the next each unit is a $K_1$-unit, where $t \in \{2, 3, 4\}$. We conclude $G$ is obtained by Steps 1, 2, $C$ of Procedure $C$. Therefore, $G \in H_{4, j}$, $j \notin \{3, 4\}$ and $(i, j) \neq (1, 4)$, i.e., $G \in H_t$ and the proof of the theorem is complete.

Thus assume for the next that $F^k_i = K^*_4$. Now we continue depending on if $v^k_i$ is a divider vertex of $K^*_4$ or non-divider vertex of $K^*_4$.

We discuss according to if \( v \in N_G(z) \setminus S_{c_1} \neq \emptyset \). Since \( c_5 \in S_{c_3} \), we have
\[
N_G(c_5) \setminus S_{c_1} \not\subseteq V(F_k^S).
\]
(9)

Suppose that \( N_G(c_5) \setminus S \subseteq V(F_k^S) \). We replace \( S \) by \( S^* = (S \setminus \{c_5\}) \cup \{c_3\} \), and do the proof in turn. We thus deduce that \( N_G(c_2) \setminus (S^*)_c \neq \emptyset \). Since \( c_3 \in (S^*)_c \), we have \( N_G(c_2) \setminus (S^*)_c \not\subseteq V(F_k^S) \), a contradiction, since with \( (S^*)_c = (S^*_{c_1} \setminus \{c_1\}) \cup \{c_3\} \) and (\( * \)), we obtain \( N_G(c_2) \setminus (S^*)_c \subseteq N_G(c_2) \setminus S_{c_1} \not\subseteq V(F_k^S) \). Then we have
\[
N_G(c_5) \setminus S \not\subseteq V(F_k^S).
\]
(10)

Set \( I_1(c_6) = N_G[c_6] \setminus N_G(x) \). If \( I_1(c_6) = \emptyset \) then \( G \in H_4 \), and \( G \) is obtained from Step 1 of Procedure \( C \). Thus assume that \( I_1(c_6) \neq \emptyset \). By the Fact 1, (9) and (10), we can use the argument of obtaining the paths \( P^{c_1}, \ldots, P^{c_k} \) as Case 1, since \( j \neq 5 \) and so by Observation 4, for the next each unit is a \( K_1 \)-unit, where \( t \in \{2, 3, 4\} \). Thus \( G \in H_4 \), and \( G \) is obtained from Steps 1, \( \ldots, k \) of Procedure \( C \) and the proof of the theorem is complete.

Next assume that \( v \) is a divider vertex. Without loss of generality, we can suppose \( v_\frac{k}{2} = c_1 \), \( v = c_6 \). By (\( * \)), \( N_G(c_6) \setminus S_{c_1} \neq \emptyset \) and \( N_G(c_6) \setminus S_{c_6} \not\subseteq V(F_k^S) \). \( N_G(c_6) \setminus V(F_k^S) = \{c_1, c_6\} \). But \( c_1 \in S_{c_1} \setminus S_{c_6} \). Hence, \( c_5 \not\in S \). Therefore, \( \{c_3, c_4\} \subseteq S \), since \( S \) covers \( c_3c_5 \) and \( c_4c_5 \). Then \( \{c_3, c_4\} \subseteq S_{c_1} \). On the other hand, for each vertex \( z \in N_G(c_1) \cap V(F_k^S) = \{c_2, c_6\} \), \( N_G(z) \setminus S_{c_1} \neq \emptyset \). Since \( c_4 \in S_{c_2} \), we have
\[
N_G(c_2) \setminus S_{c_1} \not\subseteq V(F_k^S).
\]
(11)

Suppose that there is \( t \in \{3, 4\} \) that \( N_G(c_t) \setminus S \subseteq V(F_k^S) \). We replace \( S \) by \( S^* = (S \setminus \{c_t\}) \cup \{c_3\} \), and do the proof in turn. We thus deduce that \( N_G(c_3) \setminus (S^*)_c \neq \emptyset \). Since \( c_3 \in (S^*)_c \), we have \( N_G(c_3) \setminus (S^*)_c \not\subseteq V(F_k^S) \), a contradiction, since with \( (S^*)_c = (S^*_{c_1} \setminus \{c_1\}) \cup \{c_3\} \) and (\( * \)), we obtain \( N_G(c_3) \setminus (S^*)_c \subseteq N_G(c_3) \setminus S_{c_1} \not\subseteq V(F_k^S) \). Then for \( t \in \{3, 4\} \), we have
\[
N_G(c_t) \setminus S \not\subseteq V(F_k^S).
\]
(12)

Set \( I_1(c_2) = N_G[c_2] \setminus N_G(x) \). If \( I_1(c_2) = \emptyset \) then \( G \in H_4 \), and \( G \) is obtained from Step 1 of Procedure \( C \). Thus assume that \( I_1(c_2) \neq \emptyset \). By the Fact 1, (11) and (12), we can use the argument of obtaining the paths \( P^2, \ldots, P^{k_2} \) as Case 1, since \( j \neq 5 \) and so by Observation 4, for the next each unit is a \( K_1 \)-unit, where \( t \in \{2, 3, 4\} \). Thus \( G \in H_4 \), and \( G \) is obtained from Steps 1, \( \ldots, k_2 \) of Procedure \( C \) and the proof of the theorem is complete.

Subcase 2.2. \( v_\frac{k}{2} \) is a non-divider vertex of \( K_1^*_t \). Assume that \( v_\frac{k}{2} \) has a divider neighbor. We discuss according to if \( v \), described in (\( * \)), is a divider vertex or not. First assume that \( v \) is not a divider vertex. Without loss of generality, we can assume that \( v_\frac{k}{2} = c_2 \) and \( v = c_3 \). By (\( * \)), \( N_G(c_3) \setminus S_{c_2} \neq \emptyset \) and \( N_G(c_3) \setminus S_{c_3} \not\subseteq V(F_k^S) \). \( N_G(c_3) \cap V(F_k^S) = \{c_2, c_4, c_5\} \). But \( c_2 \in S_{c_2} \). If \( c_4 \not\in S_{c_2} \), then \( c_4 \not\in S \). But \( c_2 \not\in S \), a contradiction, since \( S \) covers \( c_2c_4 \). Therefore, \( c_4 \in S_{c_2} \). Then \( c_5 \not\in S_{c_2} \). Hence, \( c_5 \not\in S \). Therefore, \( c_5 \in S \), since \( S \) covers \( c_5c_6 \). Then \( c_6 \in S_{c_1} \). On the other
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hand, for each vertex $z \in N_G(c_2) \cap V(F_k^c) = \{c_1, c_3, c_4\}$, $N_G(z) \subseteq S_c^k \neq \emptyset$. Since $c_6 \in S_c^k$, we have

$$N_G(c_1) \setminus S_c^k \not\subseteq V(F_k^c).$$

(13)

Suppose that $N_G(c_6) \setminus S \subseteq V(F_k^c)$. We replace $S$ by $S^* = (S \setminus \{c_6\}) \cup \{c_5\}$, and do the proof in turn. We thus deduce that $N_{G_1}(c_3) \setminus (S^*)_{c_6}^k \neq \emptyset$. Since $c_5 \in (S^*)_{c_5}^k$, we have $N_{G_1}(c_3) \setminus (S^*)_{c_5}^k \not\subseteq V(F_k^c)$, a contradiction, since with $(S^*)_{c_5}^k = (S_{c_5}^k \setminus \{c_6\}) \cup \{c_5\}$ and (*), we obtain $N_{G_1}(c_3) \setminus (S^*)_{c_5}^k \subseteq N_{G_1}(c_3) \setminus S_{c_5}^k \subseteq V(F_k^c)$. Then we have

$$N_G(c_1) \setminus S \not\subseteq V(F_k^c).$$

(14)

Set $I_1(c_1) = N_G(c_1) \setminus N_G(x)$. If $I_1(c_1) = \emptyset$ then $G \not\in \mathcal{H}_t$, and $G$ is obtained from Step 1 of Procedure $C$. Thus assume that $I_1(c_1) \neq \emptyset$. By the Fact 1, (13) and (14), we can use the argument of obtaining the paths $P^2, \ldots, P^k_c$ as Case 1, since $j \neq 5$ and so by Observation 4, for the next each unit is a $K_t$-unit, where $t \in \{2, 3, 4\}$. Thus $G \in \mathcal{H}_t$, and $G$ is obtained from Steps 1, $\ldots, k_G$ of Procedure $C$ and the proof of the theorem is complete.

We next assume that $v$ is a divider vertex. Without loss of generality, we can assume that $v_{k}^c = c_2$ and $v = c_1$. By (*), $N_G(c_1) \setminus S_{c_2}^k \neq \emptyset$ and $N_G(c_1) \setminus S_{c_1}^k \not\subseteq V(F_k^c)$. $N_G(c_1) \cap V(F_k^c) = \{c_2, c_6\}$. But $c_2 \in S_{c_2}^k$. Then $c_6 \not\in S_{c_2}^k$. Hence, $c_6 \not\in S$. Therefore, $c_5 \in S$, since $S$ covers $c_5c_6$. Then $c_5 \in S_{c_1}^k$, for $t \in \{3, 4\}$. On the other hand, for each vertex $z \in N_G(c_2) \cap V(F_k^c) = \{c_1, c_3, c_4\}$, $N_G(z) \subseteq S_{c_2}^k \neq \emptyset$. Since $c_2, c_3, c_5 \subseteq S_{c_2}^k$ and $c_2, c_4, c_5 \subseteq S_{c_2}^k$, for $t \in \{3, 4\}$, we have

$$N_G(c_1) \setminus S_{c_2}^k \neq V(F_k^c).$$

(15)

Suppose that $N_G(c_5) \setminus S \not\subseteq V(F_k^c)$. We replace $S$ by $S^* = (S \setminus \{c_5\}) \cup \{c_6\}$, and do the proof in turn. We thus deduce that $N_{G_1}(c_3) \setminus (S^*)_{c_5}^k \neq \emptyset$. Since $c_6 \in (S^*)_{c_6}^k$, we have $N_{G_1}(c_3) \setminus (S^*)_{c_6}^k \not\subseteq V(F_k^c)$, a contradiction, since with $(S^*)_{c_6}^k = (S_{c_6}^k \setminus \{c_6\}) \cup \{c_6\}$ and (*), we obtain $N_{G_1}(c_3) \setminus (S^*)_{c_6}^k \subseteq N_{G_1}(c_3) \setminus S_{c_6}^k \subseteq V(F_k^c)$. Then we have

$$N_G(c_5) \setminus S \not\subseteq V(F_k^c).$$

(16)

Set $I_1(c_3) = N_G(c_3) \setminus N_G(x)$. If $I_1(c_3) = \emptyset$ then $G \not\in \mathcal{H}_t$, and $G$ is obtained from Step 1 of Procedure $C$. Thus assume for the next that $I_1(c_3) \neq \emptyset$. By the Fact 1, (15) and (16), we can use the argument of obtaining the paths $P^2, \ldots, P^k_c$ as Case 1, since $j \neq 5$ and so by Observation 4, for the next each unit is a $K_t$-unit, where $t \in \{2, 3, 4\}$. Thus $G \in \mathcal{H}_t$, and $G$ is obtained from Steps 1, $\ldots, k_G$ of Procedure $C$ and the proof of the theorem is complete.

It remains to be assumed that $v_{k}^c$ has no divider neighbor. Without loss of generality, assume that $v_{k}^c = c_3$. Since $c_2, c_5 \subseteq S$, then $c_2, c_3 \subseteq S_{c_2}^k$. Thus, $N_G(c_4) \cap V(F_k^c) = \{c_2, c_3, c_5\} \subseteq S_{c_2}^k$. Then $N_G(c_4) \setminus S_{c_2}^k \not\subseteq V(F_k^c)$. Therefore, the vertex $v$ described in (*), is a neighbor of $c_3$ that is a non-divider vertex but it has a divider neighbor. Without loss of generality, suppose that $v = c_2$. By (*), $N_{G_1}(c_2) \setminus S_{c_2}^k \neq \emptyset$ and $N_{G_1}(c_2) \setminus S_{c_2}^k \not\subseteq V(F_k^c)$. $N_{G_1}(c_2) \cap V(F_k^c) = \{c_1, c_3, c_4\}$. But $c_3, c_4 \subseteq S_{c_2}^k$. Then $c_1 \not\in S_{c_2}^k$. Hence, $c_1 \not\in S$. Without loss of generality, suppose
that \( v = c_2 \). By (\( \ast \)), since \( \{c_4, c_6\} \subseteq S \), we have \( \{c_4, c_6\} \subseteq S_{c_5}^3 \). On the other hand, for each vertex \( z \in N_G(c_3) \cap V(F_k^5) = \{e_2, e_4, e_5\} \), \( N_G(z) \backslash S_{c_5}^3 \neq \emptyset \). Since \( \{c_4, c_6\} \subseteq S_{c_5}^3 \), we have

\[
N_G(c_5) \setminus S_{c_5}^3 \nsubseteq V(F_k^5).
\] (17)

Suppose that \( N_G(c_6) \setminus S \subseteq V(F_k^5) \). We replace \( S \) by \( S^* = (S \setminus \{c_6\}) \cup \{c_1\} \), and do the proof in turn. We thus deduce that \( N_G(c_2) \setminus (S^*)_{c_2}^3 \neq \emptyset \). Since \( c_1 \in (S^*)_{c_2}^3 \), we have \( N_G(c_2) \setminus (S^*)_{c_2}^3 \nsubseteq V(F_k^5) \), a contradiction, since with \( (S^*)_{c_2}^3 = (S_{c_5}^3 \setminus \{c_6\}) \cup \{c_1\} \) and \( \ast \), we obtain \( N_G(c_2) \setminus (S^*)_{c_2}^3 \subseteq N_G(c_2) \setminus S_{c_5}^3 \subseteq V(F_k^5) \). Then we have

\[
N_G(c_6) \setminus S \nsubseteq V(F_k^5).
\] (18)

Set \( I_1(c_5) = N_G[c_5] \setminus N_G(x) \). If \( I_1(c_5) = \emptyset \), then \( G \in \mathcal{H}_4 \), and \( G \) is obtained from Step 1 of Procedure \( C \). Thus assume for the next that \( I_1(c_5) \neq \emptyset \). By the Fact 1, (17) and (18), we can use the argument of obtaining the paths \( P^2, \ldots, P^k \) as Case 1, since \( j \neq 5 \) and so by Observation 4, for the next each unit is a \( K_t \)-unit, where \( t \in \{2, 3, 4\} \). Thus \( G \in \mathcal{H}_4 \), and \( G \) is obtained from Steps 1, \ldots, \( k \) of Procedure \( C \) and the proof of the theorem is complete.

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Appendix

Proof of Corollary 5. Without loss of generality, let \( G \) be a graph in \( \mathcal{G}_3 \), so that it has obtained from \( k \geq 1 \) disjoint \( K_3 \)-units by adding \( k - 1 \) edges. The units of \( G \) are pairwise disjoint. Then every \( \tau(G) \)-set has at least \( \tau(F) \) vertices from every \( F \)-unit in \( G \) to cover these units. Therefore, every \( \tau(G) \)-set has at least \( 2k \) vertices, since \( \tau(K_3) = 2 \). On the other hand, it is easy to see \( n(G) = 3k \) and \( m(G) = 4k - 1 \). Then, by Theorem 2,

\[
\tau(G) = \frac{1}{3}n \left( 1 + \frac{1}{4}m \right) + \frac{3}{12} = 2k.
\]

Therefore, every \( \tau(G) \)-set has exactly \( \tau(F) \) vertices from every \( F \)-unit in \( G \). The proof for the other graphs in \( \mathcal{G} \) is similarly verified.

References

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