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RESEARCH PAPER

Graphs with Large Total 2-Rainbow Domination Number

H. Abdollahzadeh Ahangar[1] · M. Khaibari[1] · N. Jafari Rad[2] · S. M. Sheikholeslami[3]

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Abstract

Let $G=(V,E)$ be a simple graph with no isolated vertex. A 2-*rainbow dominating function* (2RDF) of G is a function f from the vertex set $V(G)$ to the set of all subsets of the set $\{1,2\}$ such that for any vertex $v\in V(G)$ with $f(v)=\emptyset$ the condition $\bigcup_{u\in N(v)} f(u)=\{1,2\}$ is fulfilled, where $N(v)$ is the open neighborhood of v. A 2-rainbow dominating function f is called a *total* 2-*rainbow dominating function (T2RDF)* if the subgraph of G induced by $\{v\in V(G)\mid f(v)\neq\emptyset\}$ has no isolated vertex. The weight of a T2RDF f is defined as $w(f)=\sum_{v\in V(G)}|f(v)|$. The *total* 2-*rainbow domination number*, $\gamma_{tr2}(G)$, is the minimum weight of a total 2-rainbow dominating function on G. In this paper, we characterize all graphs G whose total 2-rainbow domination number is equal to their order minus one.

Keywords 2-rainbow dominating function · 2-rainbow domination number · Total 2-rainbow dominating function · Total 2-rainbow domination number

Mathematics Subject Classification 05C69

1 Introduction

For notation and graph theory terminology, we in general follow Haynes et al. (1998a, b) and Henning and Yeo (2013). In this paper, we continue the study of rainbow domination number in graphs. Specifically, let G be a simple graph with vertex set $V=V(G)$, edge set $E=E(G)$ and with no isolated vertex. The order $|V|$ of G is denoted by n and size $|E|$ of G is denoted by m. For every vertex $v\in V$, the *open neighborhood* $N_G(v)=N(v)$ is the set $\{u\in V\mid uv\in E\}$ and the *closed neighborhood* of v is the set $N_G[v]=N[v]=N(v)\cup\{v\}$. The *degree* of a vertex $v\in V$ is $\deg_G(v)=\deg(v)=|V(v)|$. The *minimum* and *the maximum degree* of a graph G are denoted by $\delta=\delta(G)$ and $\Delta=\Delta(G)$, respectively. The *open neighborhood* of a set $S\subseteq V$ is the set $N(S)=\cup_{v\in S}N(v)$, and the *closed neighborhood* of S is the set $N[S]=N(S)\cup S$. A *leaf* is a vertex of degree one, a *support vertex* is a vertex adjacent to a leaf, a *weak support vertex* is a support vertex adjacent to exactly one leaf, and a *strong support vertex* is a support vertex adjacent to at least two leaves. We denote the set of leaves and support vertices of G by $L(G)$ and $S(G)$, respectively. We also denote by L_v the set of all leaves adjacent to a support vertex v. We write K_n for the *complete graph* of order n, C_n for a *cycle* of order n and P_n for a *path* of order n. The *diameter* of G, denoted by $\mathrm{diam}(G)$, is the maximum value among minimum distances between all pairs of vertices of G. The *girth* of G, denoted by $g(G)$, is the minimum length of a cycle in G. The *corona* graph $H\odot K_1$ of a graph H is a graph obtained from H by attaching a leaf to every vertex of H. For a subset S of vertices of G, we denote by $G[S]$ the subgraph induced by S.

A subset S of vertices of a graph G is a *dominating set* of G if every vertex in $V(G)-S$ has a neighbor in S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. A dominating set S in a graph with no

✉ H. Abdollahzadeh Ahangar
ha.ahangar@nit.ac.ir

N. Jafari Rad
n.jafarirad@gmail.com

S. M. Sheikholeslami
s.m.sheikholeslami@azaruniv.edu

1 Department of Mathematics, Babol Noshirvani University of Technology, Babol, Islamic Republic of Iran

2 Department of Mathematics, Shahrood University of Technology, Shahrood, Islamic Republic of Iran

3 Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, Islamic Republic of Iran

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isolated vertex is called a *total dominating set* of G if $G[S]$ has no isolated vertex. The *total domination number* $\gamma_t(G)$ is the minimum cardinality of a total dominating set of G. The literature on the subject of domination parameters in graphs, through 1998, has been surveyed in Haynes et al. (1998a, b), and the subject of total domination in graphs, through 2013, has been surveyed in Henning and Yeo (2013). Recently, Liu and Chang (2013) introduced the concept of total Roman domination in graphs albeit in a more general setting. And very recently has been studied by Abdollahzadeh Ahangar et al. (2016, 2017).

A 2-*rainbow dominating function* (2RDF) of a graph G is a function f from the vertex set $V(G)$ to the set of all subsets of the set $\{1,2\}$ such that for any vertex $v \in V(G)$ with $f(v) = \emptyset$ the condition $\cup_{u\in N(v)} f(u) = \{1,2\}$ is fulfilled, where $N(v)$ is the open neighborhood of v. The weight of a 2RDF f is defined as $w(f) = \sum_{v\in V(G)} |f(v)|$. The minimum weight of a 2-rainbow dominating function is called the 2-*rainbow domination number* of G, denoted by $\gamma_{r2}(G)$. The concept of 2-rainbow domination was introduced by Brešar et al. (2008), and has been studied by several authors, for example Chang et al. (2010), Chellali and Jafari Rad (2013), Chunling et al. (2009), Brešar and Sumenjak (2007), Dehgardi et al. (2015), Falahat et al. (2014), Meierling et al. (2011), Sheikholeslami and Volkmann (2012), Wu and Jafari Rad (2013), Wu and Xing (2010) and Xu (2009).

If f is a 2-rainbow dominating function in a graph G, then clearly $\{v \in V(G) \mid f(v) \neq \emptyset\}$ is a dominating set of G. Abdollahzadeh Ahangar et al. (2018a) considered a variant of 2-rainbow dominating functions f such that $\{v \in V(G) \mid f(v) \neq \emptyset\}$ is a total dominating set of G. They thus introduced the concept of total 2-rainbow domination in graphs. A 2-*rainbow dominating function* f is called a *total* 2-*rainbow dominating function*, or just T2RDF if the subgraph of G induced by $\{v \in V(G) \mid f(v) \neq \emptyset\}$ has no isolated vertex. The *total* 2-*rainbow domination number*, $\gamma_{tr2}(G)$, is the minimum weight of a total 2-rainbow dominating function on G. Clearly, $\gamma_{tr2}(G)$ is well defined for any graph G with no isolated vertex, since assigning $\{1\}$ to every vertex yields a total 2-rainbow dominating function and hence

$$\gamma_{tr2}(G) \leq |V(G)|. \tag{1}$$

It is shown in Abdollahzadeh Ahangar et al. (2018b) that the decision version of the total 2-rainbow domination problem is NP-complete.

A 2-rainbow dominating function f can be represented by the ordered partition $f = (V_0^f, V_1^f, V_2^f, V_{12}^f)$, where $V_0^f = \{v \mid f(v) = \emptyset\}$, $V_1^f = \{v \mid f(v) = \{1\}\}$, $V_2^f = \{v \mid f(v) = \{2\}\}$, and $V_{12}^f = \{v \mid f(v) = \{1,2\}\}$. Thus, f is a total 2-rainbow dominating function if the subgraph of G induced by $V_1^f \cup V_2^f \cup V_{12}^f$ has no isolated vertex.

Note that if $G_1, G_2, \ldots, G_s$ are the components of G, then $\gamma_{tr2}(G) = \sum_{i=1}^{s} \gamma_{tr2}(G_i)$. Hence, it would be sufficient to consider only connected graphs in the study of total 2-rainbow domination.

Our purpose in this paper is to characterize all the connected graphs G whose total 2-rainbow domination is equal to their order minus one.

We make use of the following observations in this paper.

Observation 1 If G is a connected graph of order $n \geq 4$, and G has a support vertex x with $|L_x| \geq 2$, then $\gamma_{tr2}(G) \leq n-1$.

Observation 2 If G is a connected graph of order n, different from $K_{1,3}$, and G has a support vertex x with $|L_x| \geq 3$, then $\gamma_{tr2}(G) \leq n-2$.

Observation 3 If G is a connected graph of order n and G has two support vertices x and y with $|L_x| \geq 2$ and $|L_y| \geq 2$, then $\gamma_{tr2}(G) \leq n-2$.

Observation 4 If G is a connected graph of order n with $\mathrm{diam}(G) = 2$, then $\gamma_{tr2}(G) \leq 2\delta(G) + 1$.

Proof Let x be a vertex of minimum degree $\delta(G)$ and define $f : V(G) \to \mathcal{P}(\{1,2\})$ by $f(x) = \{1\}$, $f(u) = \{1,2\}$ for $u \in N(x)$ and $f(y) = \emptyset$ if $y \notin N[x]$. Clearly, f is a T2RDF on G of weight $2\delta(G)+1$ and so $\gamma_{tr2}(G) \leq 2\delta(G)+1$. □

2 Graphs with Large Total 2-Rainbow Domination Number

Assume $\mathcal{D} = \{H \odot K_1 \mid H \text{ is a connected graph}\}$. In Abdollahzadeh Ahangar et al. (2018a), the authors characterized all graphs G for which $\gamma_{tr2}(G) = n$ as follows:

Theorem 5 (Abdollahzadeh Ahangar et al. 2018a) *Let G be a connected graph of order n. Then, $\gamma_{tr2}(G) = n$ if and only if $G \in \mathcal{D} \cup \{P_3\}$.*

Here, we characterize all connected graphs G of order $n \geq 3$ with $\gamma_{tr2}(G) = n-1$. We start with some lemmas.

Lemma 6 *Let G be a connected graph of order n with $\delta(G) \geq 2$. Then, $\gamma_{tr2}(G) = n-1$ if and only if $G = C_3$, C_4 or C_5.*

Proof One side is clear. Let $\gamma_{tr2}(G) = n-1$. If $\mathrm{diam}(G) \geq 3$ and $w_1 w_2 \ldots w_k$ is a diametral path in G, then let $u_1 \in N(w_1) - \{w_2\}$, $v_1 \in N(w_k) - \{w_{k-1}\}$, and define the function $f : V(G) \to \mathcal{P}(\{1,2\})$ by $f(u_1) = f(v_1) =$

$\{1\}$, $f(w_1) = f(w_k) = \emptyset$ and $f(w) = \{2\}$ otherwise. It is easy to verify that f is a T2RDF on G of weight $n-2$, a contradiction. Thus, $\mathrm{diam}(G) \leq 2$. If $\mathrm{diam}(G) = 1$, then G is the complete graph of order n and we deduce from $\gamma_{tr2}(K_n) = 2$ that $G = K_3$ by the assumption. Henceforth, we assume $\mathrm{diam}(G) = 2$. By Observation 4, we have $\delta(G) \geq \frac{n-2}{2}$. Let x be a vertex of minimum degree $\delta(G)$, $N(x) = \{x_1, x_2 \ldots, x_k\}$ and $X = V(G) - N[x]$. Since $\mathrm{diam}(G) = 2$ and $\delta(G) \geq \frac{n-2}{2}$, we have $1 \leq |X| \leq \frac{n}{2}$. Assume $X = \{z_1, z_2, \ldots, z_{|X|}\}$.

If $|X| = 1$ and $\delta(G) \geq 3$, then the function $f : V(G) \to \mathcal{P}(\{1,2\})$ defined by $f(x) = f(z_1) = \emptyset, f(x_1) = \{1\}, f(v) = \{2\}$ for $v \in N(x) - \{x_1\}$, is a TR2DF of G of weight $n-2$, a contradiction. If $|X| = 1$ and $\delta(G) = 2$, then clearly $G = C_4$. Let $|X| \geq 2$.

If $\Delta(G[X]) \geq 2$ and $y \in X$ has degree at least two in $G[X]$, then the function $f : V(G) \to \mathcal{P}(\{1,2\})$ defined by $f(y) = \{1\}, f(v) = \emptyset$ for $v \in N(y) \cap X$, $f(u) = \{2\}$ otherwise, is a TR2DF of G of weight at most $n-2$, a contradiction. Assume that $\Delta(G[X]) \leq 1$. Then, all components of $G[X]$ are K_1 or K_2. It follows that each vertex in X has at least $\delta(G) - 1$ neighbors in $N(x)$ and every two vertices in X have at least $\delta(G) - 2$ common neighbors in $N(x)$. If $\delta(G) \geq 4$, then the function $f : V(G) \to \mathcal{P}(\{1,2\})$ defined by $f(x_1) = f(x_2) = \{1\}, f(u) = \emptyset$ for $u \in X$ and $f(v) = \{2\}$ otherwise, is a TR2DF of G of weight at most $n-2$, a contradiction. If $G[X]$ has two isolated vertices, say z_1, z_2, then $N(z_1) = N(z_2) = N(x)$ and the function $f : V(G) \to \mathcal{P}(\{1,2\})$ defined by $f(z_1) = f(z_2) = \emptyset, f(x_1) = \{1\}$ and $f(v) = \{2\}$ otherwise, is a TR2DF of G of weight $n-2$, a contradiction. Henceforth, we assume that $\delta(G) \leq 3$ and $G[X]$ has at most one isolated vertex that implies $G[X]$ has a K_2 component, say $z_1 z_2$.

First let $\delta(G) = 3$. Then, we have $n \leq 8$ and $|X| \leq 4$. If $|X| = 4$, then the function $f : V(G) \to \mathcal{P}(\{1,2\})$ defined by $f(x_1) = f(x_2) = \{1,2\}, f(x) = f(x_3) = \{1\}$ and $f(u) = \emptyset$ for $u \in X$, is a TR2DF of G of weight $n-2$, a contradiction. We may assume, therefore, that $|X| \leq 3$. Since $\delta(G) = 3$, we may assume that $x_1 \in N(z_1) \cap N(z_2)$. Now, it is easy to verify that the function $f : V(G) \to \mathcal{P}(\{1,2\})$ defined by $f(x_1) = \{1\}$, $f(x) = f(x_2) = f(x_3) = \{2\}$ and $f(u) = \emptyset$ for $u \in X$, is a TR2DF of G of weight $n-2$ which leads to a contradiction.

Now let $\delta(G) = 2$ and let without loss of generality that $\deg(x_1) \geq \deg(x_2)$. Then, we have $n \leq 6$ and $|X| \leq 3$. If $n = 6$, then the function $f : V(G) \to \mathcal{P}(\{1,2\})$ defined by

$$f(x_1) = \{1,2\}, f(x) = \{1\}, f(x_2) = \{2\} \text{ and } f(u) = \emptyset \text{ otherwise},$$

when $\Delta(G[X]) = 0$ or $\Delta(G[X]) = 1$ and $\deg(x_1) \geq \deg(x_2)$, and by

$$f(x_1) = \{1\}, f(x) = f(z) = \emptyset \text{ for exactly one } z \in N(x_1) \cap Z, \text{ and } f(u) = \{2\} \text{ otherwise},$$

if $\Delta(G[X]) = 1$ and $\deg(x_1) = \deg(x_2)$, is a TR2DF of G of weight 4, a contradiction. Thus, $n = 5$ and so $|X| = 2$. It is easy to check that $G = C_5$ in this case and the proof is complete. □

Assume $\mathcal{D}_1$ is the family of graphs consisting of all graphs G such that G is obtained from a graph in $\mathcal{D} \cup \{P_3\}$ by adding a pendant edge to precisely one support vertex.

Lemma 7 *Let G be a graph such that each vertex of G is a leaf or a support vertex. Then, $\gamma_{tr2}(G) = n-1$ if and only if $G \in \mathcal{D}_1$.*

Proof Let $\gamma_{tr2}(G) = n-1$. Since each vertex of G is a leaf or a support vertex, it follows from Theorem 5 that G has a strong support vertex. If $G = K_{1,3}$, then clearly $G \in \mathcal{D}_1$. Let $G \neq K_{1,3}$. It follows from Observations 2 and 3 that G has exactly one support vertex which is adjacent to exactly two leaves. Thus, $G \in \mathcal{D}_1$.

Conversely, let $G \in \mathcal{D}_1$. Then, G is obtained from a graph in $\mathcal{D} \cup \{P_3\}$, by adding a pendant edge to precisely one support vertex of degree at least two. By Theorem 5, we have $\gamma_{tr2}(G) \leq n-1$. Now, we show that $\gamma_{tr2}(G) \geq n-1$. Let f be a $\gamma_{tr2}(G)$-function. If G is obtained from P_3, then $G = K_{1,3}$ and clearly $\gamma_{tr2}(G) = 3$. Assume G is obtained from $H \odot K_1$ for some connected graph H of order at least two. Let $V(H) = \{u_1, u_2 \ldots, u_m\}$, v_i be the leaf adjacent to u_i for each i, and w be the new pendant vertex joined to u_m. Since f is a TR2DF of G, we have $|f(u_i)| + |f(v_i)| \geq 2$ for each i and this implies that $\gamma_{tr2}(G) \geq 2m = n-1$ as desired. □

Regarding Lemmas 6 and 7, it is enough to consider graphs G such that $\delta(G) = 1$ and $S(G) \cup L(G) \subsetneqq V(G)$.

Lemma 8 *Let G be a connected graph of order n with $\delta(G) = 1$ and $S(G) \cup L(G) \subsetneqq V(G)$. If $\gamma_{tr2}(G) = n-1$, then the induced subgraph $G[V(G) - (S(G) \cup L(G))]$ is isomorphic to K_1, K_2, P_3, C_3, or C_4.*

Proof First, we show that $G - (S(G) \cup L(G))$ is a connected graph. Assume, to the contrary, that B_1 and B_2 are two components of $G - (S(G) \cup L(G))$ and $x_i \in B_i$ for $i = 1, 2$. Since $x_i \notin S(G) \cup L(G)$, x_i has two neighbors x_i^1, x_i^2 each of degree at least two, for $i = 1, 2$. Then, the function $f : V(G) \to \mathcal{P}(\{1,2\})$ defined by $f(x_1) = f(x_2) = \emptyset$, $f(x_1^1) = f(x_2^1) = \{2\}$ and $f(x) = \{1\}$ otherwise, is a TR2DF of G of weight at most $n-2$, a contradiction. Hence, G is connected. Consider two cases.

Case 1 $G[V(G) - (S(G) \cup L(G))]$ contains a cycle.
Assume $C = (x_1 x_2 \ldots x_k)$ is a cycle of $G[V(G) - (S(G) \cup L(G))]$. If $k \geq 6$, then the function $f : V(G) \to \mathcal{P}(\{1,2\})$ defined by $f(x_1) = f(x_2) = \{1\}, f(x_3) = f(x_k) = \emptyset$ and $f(x) = \{2\}$ otherwise, is a

TR2DF of G of weight $n-2$, a contradiction. If $k=5$, then since G is connected and $V(G) \neq S(G) \cup L(G)$, we may assume that x_1 has a neighbor w outside $V(C)$. Then, the function $f : V(G) \to \mathcal{P}(\{1,2\})$ defined by $f(x_1) = f(w) = \{1\}, f(x_2) = f(x_5) = \emptyset$ and $f(x) = \{2\}$ otherwise, is a TR2DF of G of weight at most $n-2$, a contradiction. Thus, $k \leq 4$.

First let $k=4$. If C has a chord, say x_1x_3, then the function $f = (\{x_2, x_4\}, V(G) - \{x_1, x_2, x_4\}, \{x_1\}, \emptyset)$ is a TR2DF of G of weight $n-2$, a contradiction. So C has no chord. If $G[V(G) - (S(G) \cup L(G))] \neq C$, then we may assume x_1 has a neighbor w outside $S(G) \cup L(G)$ and the function $f : V(G) \to \mathcal{P}(\{1,2\})$ defined by $f(x_4) = f(w) = \emptyset$, $f(x_1) = \{2\}$ and $f(x) = \{1\}$ otherwise, is a TR2DF of G of weight $n-2$, a contradiction. Hence, $G[V(G) - (S(G) \cup L(G))] \cong C_4$ in this case.

Now, let $k=3$. If $G[V(G) - (S(G) \cup L(G))] \neq C$, then we may assume x_1 has a neighbor w outside $S(G) \cup L(G)$ and the function $f = (\{w, x_3\}, V(G) - \{x_1, w, x_3\}, \{x_1\}, \emptyset)$ is a TR2DF of G of weight $n-2$, a contradiction. Thus, $G[V(G) - (S(G) \cup L(G))] \cong C_3$ in this case.

Case 2 $G[V(G) - (S(G) \cup L(G))]$ is a tree.
If $\Delta(G[V(G) - (S(G) \cup L(G))]) \geq 3$, then let x be a vertex of maximum degree in $G[V(G) - (S(G) \cup L(G))]$ and let y_1, y_2, y_3 be the neighbors of x in $G[V(G) - (S(G) \cup L(G))]$. Then, the function $f : V(G) \to \mathcal{P}(\{1,2\})$ defined by $f(y_2) = f(y_3) = \emptyset$, $f(x) = f(y_1) = \{2\}$ and $f(x) = \{1\}$ otherwise, is a TR2DF of G of weight $n-2$, a contradiction. Thus, $\Delta(G[V(G) - (S(G) \cup L(G))]) \leq 2$ and so $G[V(G) - (S(G) \cup L(G))]$ is a path $P_m = x_1x_2\ldots x_m$. If $m \geq 4$, then the function $f : V(G) \to \mathcal{P}(\{1,2\})$ defined by $f(x_1) = f(x_m) = \emptyset$, $f(x_i) = \{2\}$ for $2 \leq i \leq m-1$, and $f(x) = \{1\}$ otherwise, is a TR2DF of G of weight $n-2$, a contradiction. Thus, $m \leq 3$ and the proof is complete. □

Recall that $\mathcal{D} = \{H \odot K_1 \mid H \text{ is a connected graph}\}$. Now, we introduce the following families of graphs.

1. $\mathcal{D}_2$ is the family of graphs consisting of all graphs G such that G is obtained from $r \geq 2$ graphs $G_1, G_2, \ldots, G_r \in \mathcal{D}$, by adding a new vertex, called head, and joining it to a support vertex of G_i, for each $i = 1, 2, \ldots, r$.
2. $\mathcal{D}_3$ is the family of graphs consisting of all graphs G such that G is obtained from $r \geq 1$ graphs $G_1, G_2, \ldots, G_r \in \mathcal{D}$ and P_3, by adding a new vertex, called head, and joining it to the support vertex of P_3 and a support vertex of G_i, for each $i = 1, 2, \ldots, r$.
3. $\mathcal{D}_4$ is the family of graphs consisting of all graphs G such that G is obtained from $r \geq 1$ graphs $G_1, G_2, \ldots, G_r \in \mathcal{D}$, where $G_r = K_2$, by adding a path x_1x_2 and joining x_1 to a support vertex of G_i, for each $i = 1, 2, \ldots, r-1$ and x_2 to a leaf of G_r.
4. $\mathcal{D}_5$ is the family of graphs consisting of all graphs G such that G is obtained from two graphs $G_1, G_2 \in \mathcal{D}_2$, by adding a new vertex and joining it to the head of D_i for $i = 1, 2$.
5. $\mathcal{D}_6$ is the family of graphs consisting of all graphs G such that G is obtained from $r \geq 2$ graphs $G_1, G_2, \ldots, G_r \in \mathcal{D}$, by adding a path xyz, joining x to a support vertex of G_i, for each $i = 1, 2, \ldots, r-1$ and joining z to a support vertex of G_r.
6. $\mathcal{D}_7$ be a family of graphs consisting of all graphs G such that G is obtained from a cycle $(x_1x_2x_3)$ by joining x_1 to a support vertex of finitely many graphs $G_1^1, G_2^1, \ldots, G_{r_1}^1 \in \mathcal{D}$ $(r_1 \geq 1)$ and x_2 to a support vertex of finitely many graphs $G_1^2, G_2^2, \ldots, G_{r_2}^2 \in \mathcal{D}$ (possibly no graphs).
7. $\mathcal{D}_8$ be a family of graphs consisting of all graphs G such that G is obtained from a cycle $C_4 = (x_1x_2x_3x_4)$ by joining x_1 to a support vertex of finitely many graphs $G_1, G_2, \ldots, G_r \in \mathcal{D}$ $(r \geq 1)$.

Lemma 9 *Let G be a graph of order n. If $G \in \cup_{i=1}^{8} \mathcal{D}_i$, then $\gamma_{tr2}(G) = n-1$.*

Proof Let $G \in \cup_{i=1}^{8} \mathcal{D}_i$. If $G \in \mathcal{D}_1$, then $\gamma_{tr2}(T) = n-1$ by Lemma 7. Suppose $T \in \cup_{i=2}^{8} \mathcal{D}_i$. By Theorem 5, we have $\gamma_{tr2}(G) \leq n-1$. Now, we show that $\gamma_{tr2}(G) \geq n-1$. Suppose f is a $\gamma_{tr2}(G)$-function.

Let $G \in \mathcal{D}_2$. Then, G is obtained from $r \geq 2$ graphs $H_1 \odot K_1, H_2 \odot K_1, \ldots, H_r \odot K_1$ for each i, by adding a new vertex and joining it to a support vertex of H_i for $i = 1, 2, \ldots, r$. Let $V(H_i) = \{u_1^i, u_2^i \ldots, u_{m_i}^i\}$ and v_j^i be the leaf adjacent to u_j^i for each i and j. Clearly, $|f(u_j^i)| + |f(v_j^i)| \geq 2$ i, j and so $\gamma_{tr2}(G) = \omega(f) \geq \sum_{i=1}^{r} 2|V(H_i)| = n-1$ as desired.

Let $G \in \mathcal{D}_3$. Then, G is obtained from $r \geq 1$ graphs $H_1 \odot K_1, H_2 \odot K_1, \ldots, H_r \odot K_1$ for each i, by adding a new vertex w and joining it to the support vertex of a path $P_3 = z_1z_2z_3$ and to a vertex of H_i, for each $i = 1, 2, \ldots, r$. Let $V(H_i) = \{u_1^i, u_2^i, \ldots, u_{m_i}^i\}$ and v_j^i be the leaf adjacent to u_j^i for each i and j. Clearly, $|f(w)| + |f(z_1)| + |f(z_2)| + |f(z_3)| \geq 3$ and $|f(u_j^i)| + |f(v_j^i)| \geq 2$ for each i, j and so $\gamma_{tr2}(G) = \omega(f) \geq 3 + \sum_{i=1}^{r} 2|V(H_i)| = n-1$ as desired.

Let $G \in \mathcal{D}_4$. Then, G is obtained from $r \geq 2$ graphs $H_1 \odot K_1, H_2 \odot K_1, \ldots, H_{r-1} \odot K_1$ and $G_r = K_2 = z_1z_2$ by adding a path x_1x_2, joining x_2 to z_2 and x_1 to a vertex of H_i, for each $i = 1, 2, \ldots, r-1$. Let $V(H_i) = \{u_1^i, u_2^i, \ldots, u_{m_i}^i\}$ and v_j^i be the leaf adjacent to u_j^i for each i and j. Clearly, $|f(x_1)| + |f(x_2)| + |f(z_1)| + |f(z_2)| \geq 3$ and as above we have $\gamma_{tr2}(T) = \omega(f) \geq \sum_{i=1}^{r} 2|V(H_i)| = n-1$ as desired.

Let $G \in \mathcal{D}_5$. Then, G is obtained from graphs $H_1^1 \odot K_1, H_2^1 \odot K_1, \ldots, H_{m_1}^1 \odot K_1, H_1^2 \odot K_1, H_2^2 \odot K_1, \ldots, H_{m_2}^2 \odot K_1$, with $m_1, m_2 \geq 2$, by adding a path $P_3 = x_1x_2x_3$, and joining x_i to a support vertex of H_j^i, for each j and $i = 1, 3$. To totally rainbowly dominate x_2 we must have $|f(x_1)| + |f(x_2)| + |f(x_3)| \geq 2$ and as above case we obtain

$$\begin{aligned}\gamma_{tr2}(G) &= \omega(f)\\ &\geq 3 + \sum_{i=1}^{2}\sum_{j=1}^{m_i}\left(\sum_{x\in V(H_j^i\odot K_1)}|f(x)|\right)\\ &= 3 + \sum_{i=1}^{2}\sum_{j=1}^{m_i}|V(H_j^i \odot K_1)|\\ &\geq n-1,\end{aligned}$$

as desired. The proofs for the cases $G \in \mathcal{D}_6$, $G \in \mathcal{D}_7$ and $G \in \mathcal{D}_8$ are similar. □

Lemma 10 *Let G be a connected graph of order n with $\delta(G) = 1$ and $L(G) \cup S(G) \subsetneqq V(G)$. Then, $\gamma_{tr2}(G) = n - 1$ if and only if $G \in \cup_{i=2}^{8}\mathcal{D}_i$.*

Proof Sufficiency is true by Lemma 9. It is enough to prove necessity. Let $\gamma_{tr2}(G) = n - 1$. Clearly, $n \geq 4$. If $n = 4$, then $G = K_{1,3} \in \mathcal{D}_1$. Assume $n \geq 5$. If $V(G) = L(G) \cup S(G)$, then $G \in \mathcal{D}_1$ by Lemma 7. Let $L(G) \cup S(G) \subsetneqq V(G)$. By Lemma 8, $G[V(G) - (S(G) \cup L(G))]$ is isomorphic to K_1, P_2, P_3, C_3, or C_4. We consider the following cases.

Case 1 $G[V(G) - (S(G) \cup L(G))] \cong K_1 = w$.

Let $G_1, G_2, \ldots, G_k$ be the components of $G - w$. We deduce from $G[V(G) - (S(G) \cup L(G))] = w$ that $k \geq 2$, $|V(G_i)| \geq 2$ for each i, and w is adjacent to a support vertex w_i of G_i for each i. It follows that G_i has a γ_{tr2}-function g_i such that $1 \in g_i(w_i)$ for each i. If $\gamma_{tr2}(G_i) \leq |V(G_i)| - 1$ for some i, say $i = 1$, then the function $f : V(G) \to \mathcal{P}(\{1,2\})$ defined by $f(w) = \emptyset, f(u) = g_1(u)$ for $u \in V(G_1)$, and $f(u) = \{2\}$ otherwise, is a TR2DF of T of weight $n - 2$, a contradiction. Thus, $\gamma_{tr2}(G_i) = |V(G_i)|$ for each i. This implies that $G_i \in \mathcal{D} \cup P_3$ for $1 \leq i \leq k$. If $G_i = P_3 = y_1w_iy_3$ and $G_j = P_3 = y_1'w_jy_3'$ for some $i \neq j$, then the function $f : V(G) \to \mathcal{P}(\{1,2\})$ defined by $f(y_1) = f(y_3) = f(y_1') = f(y_3') = \emptyset, f(w_i) = f(w_j) = \{1,2\}, f(w) = 1$, and $f(u) = \{1\}$ otherwise, is a TR2DF of T of weight at most $n - 2$, a contradiction. Thus at most one of G_i's is P_3. This implies that $G \in \mathcal{D}_2 \cup \mathcal{D}_3$.

Case 2 $G[V(G) - (S(G) \cup L(G))] \cong P_2 = x_1x_2$.

Let $G_1^i, G_2^i, \ldots, G_{m_i}^i$ be the components of $G - \{x_1, x_2\}$ adjacent to x_i for $i = 1, 2$. Since x_1 and x_2 are not support vertices, $|V(G_j^i)| \geq 2$ for $i = 1, 2$ and each j. If $m_1, m_2 \geq 2$, then the function $f : V(G) \to \mathcal{P}(\{1,2\})$ defined by $f(x_1) = f(x_2) = \emptyset, f(u) = \{1\}$ for $u \in V(G_1^1) \cup V(G_1^2)$, and $f(u) = \{2\}$ otherwise, is a TR2DF of T of weight $n - 2$, a contradiction. Assume that $m_2 = 1$. Since x_1, x_2 are the only vertices of G which are not leaf or support vertex, x_1 is adjacent to a support vertex of G_j^1 for each $1 \leq j \leq m_1$ and x_2 is adjacent to a support vertex w_2 of G_1^2. If $\gamma_{tr2}(G_1^2) \leq |V(G_1^2)| - 1$, then let g be a $\gamma_{tr2}(G_1^2)$-function so that $1 \in g(w_2)$ and define $f : V(G) \to \mathcal{P}(\{1,2\})$ by $f(x_2) = \emptyset, f(u) = g(u)$ for $u \in V(G_1^2)$, and $f(u) = \{2\}$ otherwise. Clearly, f is a TR2DF of T of weight at most $n - 2$, a contradiction. Thus, $\gamma_{tr2}(G_1^2) = |V(G_1^2)|$. It is easy to verify that $G_1^2 \neq P_3$ and so $G_1^2 \in \mathcal{D}$.

If $G_1^2 = w_2w_2'$, then as above we can see that $\gamma_{tr2}(G_j^1) = |V(G_j^1)|$ for each j, and so $G \in \mathcal{D}_4$. Let $|V(G_1^2)| \geq 4$. Then, G_1^2 has a γ_{tr2}-function g_2 so that $g_2(w_2) = \{1,2\}$. If $m_1 \geq 2$, then the function $f : V(G) \to \mathcal{P}(\{1,2\})$ defined by $f(x_1) = f(x_2) = \emptyset, f(u) = g_2(u)$ for $u \in V(G_1^2), f(u) = \{1\}$ for $u \in V(G_1^1)$ and $f(u) = \{2\}$ otherwise, is a TR2DF of T of weight $n - 2$, a contradiction. Therefore, $m_1 = 1$. As above, we can see that $\gamma_{tr2}(G_1^1) = |V(G_1^1)|$ and $G_1^1 \neq P_3$. Let x_1 be adjacent to the support vertex w_1 of G_1^1. If $|V(G_1^1)| \geq 4$, then let g_1 be a $\gamma_{tr2}(G_1^1)$-function such that $g_1(w_1) = \{1,2\}$ and define the function $f : V(G) \to \mathcal{P}(\{1,2\})$ by $f(x_1) = f(x_2) = \emptyset, f(u) = g_1(u)$ for $u \in V(G_1^1)$, and $f(u) = g_2(u)$ for $u \in V(G_1^2)$. Then, f is a TR2DF of T of weight at most $n - 2$ which is a contradiction. Thus, $|V(G_1^1)| = 2$ that implies $G \in \mathcal{D}_4$.

Case 3 $G[V(G) - (S(G) \cup L(G))] \cong P_3 = x_1x_2x_3$.

If $\deg(x_2) \geq 3$, then the function $f : V(G) \to \mathcal{P}(\{1,2\})$ defined by $f(x_1) = f(x_3) = \emptyset, f(x_2) = \{1\}$, and $f(u) = \{2\}$ otherwise, is a TR2DF of T of weight $n - 2$, a contradiction. Hence, $\deg(x_2) = 2$. Assume $G_1^i, G_2^i, \ldots, G_{m_i}^i$ are the components of $G - \{x_1, x_2, x_3\}$ adjacent to x_i for $i = 1, 3$. Since x_1 and x_3 are not support vertices, $|V(G_j^i)| \geq 2$ for each i and $j = 1, 3$. If $\gamma_{tr2}(G_j^1) \leq |V(G_j^1)| - 1$ for some j, then let g be a $\gamma_{tr2}(G_j^1)$-function and define $f : V(G) \to \mathcal{P}(\{1,2\})$ by $f(x_3) = \emptyset, f(x_2) = \{1\}, f(u) = g(u)$ for $u \in V(G_j^1)$, and $f(u) = \{2\}$ otherwise. Clearly, f is a TR2DF of T of weight at most $n - 2$, a contradiction. Thus, $\gamma_{tr2}(G_j^1) = |V(G_j^1)|$ for each $1 \leq i \leq j \leq m_1$. If $G_j^1 = P_3 = y_1y_2y_3$ for some j, then define $f : V(G) \to \mathcal{P}(\{1,2\})$ by $f(y_1) = f(y_3) = f(x_2) = \emptyset, f(y_2) = \{1,2\}, f(x_1) = \{1\}$ and $f(u) = \{2\}$ otherwise, when $x_1y_2 \in E(G)$, and by $f(y_1) = f(x_3) = \emptyset, f(x_1) = \{1\}$ and $f(u) = \{2\}$ otherwise, when $x_1y_1 \in E(G)$. Clearly, f is a TR2DF of T of weight at most $n - 2$, a contradiction. Thus, $G_j^1 \neq P_3$ and so $G_j^1 \in \mathcal{D}$ for each j. Similarly, $G_j^2 \in \mathcal{D}$ for each $1 \leq j \leq m_3$. If $|V(G_j^1)| \geq 4$ for some j, and x_1 is adjacent to a leaf of G_j^1 such as v, then the function $f : V(G) \to \mathcal{P}(\{1,2\})$ defined by $f(x_3) = f(v) = \emptyset, f(x_1) = f(x_2) = \{1\}$, and $f(u) = \{2\}$ otherwise, is a TR2DF of T of weight at most $n-2$, a contradiction. Thus, x_1 is adjacent to a support vertex of G_j^1 for $1 \le j \le m_1$. Similarly, x_3 is adjacent to a support vertex of G_j^2 for each $1 \le j \le m_3$. Thus, $G \in \mathcal{D}_6$ if $m_1, m_2 \ge 2$ and $G \in \mathcal{D}_7$ if $m_1 = 1$ or $m_2 = 1$.

Case 4 $G[V(G) - (S(G) \cup L(G))] \cong C_3 = (x_1x_2x_3)$.
If $\deg(x_i) \ge 3$ for $i = 1, 2, 3$, then the function $f : V(G) \to \mathcal{P}(\{1,2\})$ defined by $f(x_1) = f(x_2) = \emptyset, f(x_3) = \{1\}$, and $f(u) = \{2\}$ otherwise, is a TR2DF of T of weight at most $n-2$, a contradiction. Henceforth, we may assume that $\deg(x_3) = 2$. Since G is connected and $\delta(G) = 1$, we may assume that $\deg(x_1) \ge 3$. Let $G_1, G_2, \ldots, G_k$ be the components of $G - \{x_1, x_2, x_3\}$. Since x_1, x_2, x_3 are not support vertices, we have $|V(G_i)| \ge 2$ for each i. If $\gamma_{tr2}(G_i) \le |V(G_i)| - 1$ for some i, say $i = 1$, then let g be a $\gamma_{tr2}(G_1)$-function and define $f : V(G) \to \mathcal{P}(\{1,2\})$ by $f(x_1) = \emptyset, f(x_2) = \{1\}, f(x_3) = \{2\}, f(u) = g(u)$ for $u \in V(G_1)$, and $f(u) = \{1\}$ otherwise. Obviously, f is a TR2DF of T of weight at most $n-2$, a contradiction. Thus, $\gamma_{tr2}(G_i) = |V(G_i)|$ for each i. As in Case 3, we can see that $G_i \in \mathcal{D}$ for each i. If $|V(G_i)| \ge 4$ for some i, say $i = 1$, and x_1 is adjacent to a leaf of G_1 such as v, then the function $f : V(G) \to \mathcal{P}(\{1,2\})$ defined by $f(x_3) = f(v) = \emptyset$, $f(x_1) = \{1\}$, and $f(u) = \{2\}$ otherwise, is a TR2DF of T of weight at most $n-2$, a contradiction. Thus, x_1 is adjacent to a support vertex of some G_i. Similarly, x_2 is adjacent to a support vertex of some G_j if $\deg(x_2) \ge 3$. It follows that $G \in \mathcal{D}_8$.

Case 5 $G[V(G) - (S(G) \cup L(G))] \cong C_4 = (x_1x_2x_3x_4)$.
Let $G_1, G_2, \ldots, G_k$ be the components of $G - \{x_1, x_2, x_3, x_4\}$. Since x_1, x_2, x_3, x_4 are not support vertices, $|V(G_i)| \ge 2$ for each i. If $\gamma_{tr2}(G_i) \le |V(G_i)| - 1$ for some i, say $i = 1$, then let g be a $\gamma_{tr2}(G_1)$-function and define $f : V(G) \to \mathcal{P}(\{1,2\})$ by $f(x_1) = \emptyset, f(x_2) = \{1\}, f(x_3) = f(x_4) = \{2\}, f(u) = g(u)$ for $u \in V(G_1)$, and $f(u) = \{1\}$ otherwise. Clearly, f is a TR2DF of T of weight at most $n-2$ which is a contradiction. Thus, $\gamma_{tr2}(G_i) = |V(G_i)|$ for each i. As in Case 3, we can see that $G_i \in \mathcal{D}$ for each i. If $\deg(x_1) \ge 3, \deg(x_2) \ge 3$, then the function $f : V(G) \to \mathcal{P}(\{1,2\})$ defined by $f(x_1) = f(x_2) = \emptyset, f(x_3) = f(x_4) = \{1\}$, and $f(u) = \{2\}$ otherwise, is a TR2DF of T of weight at most $n-2$, a contradiction. If $\deg(x_1) \ge 3, \deg(x_3) \ge 3$, then the function $f : V(G) \to \mathcal{P}(\{1,2\})$ defined by $f(x_4) = f(x_2) = \emptyset, f(x_1) = \{1\}$, $f(x_3) = \{2\}$, and $f(u) = \{2\}$ otherwise, is a TR2DF of T of weight at most $n-2$, a contradiction. Henceforth, exactly one of the x_is ($i = 1, 2, 3$) has degree greater than 2. Assume that $\deg(x_1) \ge 3$. If $|V(G_i)| \ge 4$ for some i, say $i = 1$, and x_1 is adjacent to a leaf of G_1 such as v, then the function $f : V(G) \to \mathcal{P}(\{1,2\})$ defined by $f(x_4) = f(v) = \emptyset, f(x_3) = \{1\}$, and $f(u) = \{2\}$ otherwise, is a TR2DF of T of weight at most $n-2$, a contradiction. Thus, x_1 is adjacent to a support vertex of each G_i. This implies that $G \in \mathcal{D}_8$. This completes the proof. □

Considering Lemmas 6 and 11, we are now ready to state the main theorem of this paper.

Theorem 11 *Let G be a connected graph of order n. Then, $\gamma_{tr2}(G) = n-1$ if and only if $G \in \{C_3, C_4, C_5\} \cup (\cup_{i=1}^{8} \mathcal{D}_i)$.*

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References

Abdollahzadeh Ahangar H, Amjadi J, Sheikholeslami SM, Soroudi M (2017) On the total Roman domination number of graph. Ars Combin (**to appear**)

Abdollahzadeh Ahangar H, Henning MA, Samodivkin V, Yero IG (2016) Total Roman domination in graphs. Appl Anal Discret Math 10:501–517

Abdollahzadeh Ahangar H, Amjadi J, Jafari Rad N, Samodivkin V (2018a) Total k-rainbow domination numbers in graphs. Commun Comb Optim 3:37–50

Abdollahzadeh Ahangar H, Amjadi J, Chellali M, Nazari-Moghaddam S, Shahbazi L, Sheikholeslami SM (2018b) Total 2-rainbow domination numbers in trees (**submitted**)

Chang GJ, Wu J, Zhu X (2010) Rainbow domination on trees. Discret Appl Math 158:8–12

Chellali M, Jafari Rad N (2013) On 2-rainbow domination and Roman domination in graphs. Australas J Comb 56:85–93

Chunling T, Xiaohui L, Yuansheng Y, Meiqin L (2009) 2-rainbow domination of generalized Petersen graphs $P(n; 2)$. Discret Appl Math 157:1932–1937

Brešar B, Henning MA, Rall DF (2008) Rainbow domination in graphs. Taiwan J Math 12:201–213

Brešar B, Sumenjak TK (2007) On the 2-rainbow domination in graphs. Discret Appl Math 155:2394–2400

Dehgardi N, Sheikholeslami SM, Volkmann L (2015) The rainbow domination subdivision numbers of graphs. Mat Vesnik 67:102–114

Falahat M, Sheikholeslami SM, Volkmann L (2014) New bounds on the rainbow domination subdivision number. Filomat 28:615–622

Haynes TW, Hedetniemi ST, Slater PJ (1998) Fundamentals of domination in graphs. Marcel Dekker, New York

Haynes TW, Hedetniemi ST, Slater PJ (1998) Domination in graphs: advanced topics. Marcel Dekker Inc., New York

Henning MA, Yeo A (2013) Total domination in graphs. New York, Springer

Liu C-H, Chang GJ (2013) Roman domination on strongly chordal graphs. J Comb Optim 26:608–619

Meierling D, Sheikholeslami SM, Volkmann L (2011) Nordhaus–Gaddum bounds on the k- rainbow domatic number of a graph. Appl Math Lett 24:1758–1761

Sheikholeslami SM, Volkmann L (2012) The k -rainbow domatic number of a graph. Discuss Math Graph Theory 32:129–140

Wu Y, Jafari Rad N (2013) Bounds on the 2-rainbow domination number of graphs. Graphs Comb 29:1125–1133

Wu Y, Xing H (2010) Note on 2-rainbow domination and Roman domination in graphs. Appl Math Lett 23:706–709

Xu G (2009) 2-rainbow domination of generalized Petersen graphs $P(n; 3)$. Discret Appl Math 157:2570–2573