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BOUNDS ON THE LOCATING ROMAN DOMINATION NUMBER IN TREES

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Abstract

A Roman dominating function (or just RDF) on a graph $G = (V, E)$ is a function $f : V \longrightarrow \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of an RDF f is the value $f(V(G)) = \sum_{u \in V(G)} f(u)$. An RDF f can be represented as $f = (V_0, V_1, V_2)$, where $V_i = \{v \in V : f(v) = i\}$ for $i = 0, 1, 2$. An RDF $f = (V_0, V_1, V_2)$ is called a locating Roman dominating function (or just LRDF) if $N(u) \cap V_2 \neq N(v) \cap V_2$ for any pair u, v of distinct vertices of V_0. The locating Roman domination number $\gamma_R^L(G)$ is the minimum weight of an LRDF of G. In this paper, we study the locating Roman domination number in trees. We obtain lower and upper bounds for the locating Roman domination number of a tree in terms of its order and the number of leaves and support vertices, and characterize trees achieving equality for the bounds.

Keywords: Roman domination number, locating domination number, locating Roman domination number, tree.

2010 Mathematics Subject Classification: 05C69.

1. Introduction

In this paper, we continue the study of a variant of Roman dominating functions, namely, locating Roman dominating functions introduced in [16]. We first present some necessary definitions and notations. For notation and graph theory terminology not given here, we follow [13]. We consider finite, undirected, and simple graphs G with vertex set $V = V(G)$ and edge set $E = E(G)$. The number of vertices of a graph G is called the *order* of G and is denoted by $n = n(G)$. The *open neighborhood* of a vertex $v \in V$ is $N(v) = N_G(v) = \{u \in V : uv \in E\}$, and the *degree* of v, denoted by $\deg_G(v)$, is the cardinality of its open neighborhood. A *leaf* of a tree T is a vertex of degree one, while a *support vertex* of T is a vertex adjacent to a leaf. A *strong support vertex* is a support vertex adjacent to at least two leaves. In this paper, we denote the set of all strong support vertices of T by $S(T)$ and the set of leaves by $L(T)$. We denote $\ell(T) = |L(T)|$ and $s(T) = |S(T)|$. We also denote by $L(x)$ the set of leaves adjacent to a support vertex x, and denote $\ell_x = |L(x)|$. If T is a rooted tree then for any vertex v we denote by T_v the subtree rooted at v. A subset $S \subseteq V$ is a *dominating set* of G if every vertex in $V - S$ has a neighbor in S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G.

The study of locating dominating sets in graphs was pioneered by Slater [21, 22]. For many problems related to graphs, various types of protection sets are studied where the objective is to precisely locate an "intruder". It is considered that a detection device at a vertex v is able to determine if the intruder is at v or if it is in $N(v)$, but at which vertex in $N(v)$, it cannot be determined. A *locating-dominating set* $D \subseteq V(G)$ is a dominating set with the property that for each vertex $x \in V(G) - D$ the set $N(x) \cap D$ is unique. That is, any two vertices x, y in $V(G) - D$ are distinguished in the sense that there is a vertex $v \in D$ with $|N(v) \cap \{x, y\}| = 1$. The minimum size of a locating-dominating set for a graph G is the *locating-domination number* of G, denoted $\gamma_L(G)$. The concept of locating domination has been considered for several domination parameters, see for example [4, 5, 6, 8, 9, 11, 12, 14, 15, 18, 23].

For a graph G, let $f : V(G) \to \{0, 1, 2\}$ be a function, and let (V_0, V_1, V_2) be the ordered partition of $V(G)$ induced by f, where $V_i = \{v \in V(G) : f(v) = i\}$ for $i = 0, 1, 2$. There is a $1-1$ correspondence between the functions $f : V(G) \to \{0, 1, 2\}$ and the ordered partitions (V_0, V_1, V_2) of $V(G)$. So we will write $f = (V_0, V_1, V_2)$. A function $f : V(G) \to \{0, 1, 2\}$ is a *Roman dominating function* (or just RDF) if every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of an RDF f is $w(f) = f(V(G)) = \sum_{u \in V(G)} f(u)$. The *Roman domination number* of a graph G, denoted by $\gamma_R(G)$, is the minimum weight of an RDF on G. A function $f = (V_0, V_1, V_2)$ is called a *γ_R-function* (or *$\gamma_R(G)$-function* when we want to refer f to G), if it is an RDF and $f(V(G)) = \gamma_R(G)$, see [10, 19, 24].

Roman dominating functions with several further conditions have been studied, for example, among other types, see for example [1, 2, 3, 7, 17, 20].

It is known [10] that if $f = (V_0, V_1, V_2)$ is an RDF in a graph G then V_2 is a dominating set for $G[V_0 \cup V_2]$. Jafari Rad, Rahbani and Volkmann [16] considered Roman dominating functions $f = (V_0, V_1, V_2)$ with a further condition that for each vertex $x \in V_0$ the set $N(x) \cap V_2$ is unique. That is, any two vertices x, y in V_0 are distinguished in the sense that there is a vertex $v \in V_2$ with $|N(v) \cap \{x, y\}| = 1$.

An RDF $f = (V_0, V_1, V_2)$ is called a *locating Roman dominating function* (or just LRDF) if $N(v) \cap V_2 \neq N(u) \cap V_2$ for any pair u, v of distinct vertices of V_0. The *locating Roman domination number* $\gamma_R^L(G)$ is the minimum weight of an LRDF. Note that $\gamma_R^L(G)$ is defined for any graph G, since $(\emptyset, V(G), \emptyset)$ is an LRDF for G. We refer to a $\gamma_R^L(G)$*-function* as an LRDF of G with minimum weight. It is shown in [16] that the decision problem for the locating Roman domination problem is NP-complete for bipartite graphs and chordal graphs. Moreover, several bounds and characterizations are given for the locating Roman domination number of a graph.

In this paper we study the locating Roman domination number in trees. In Section 2, we show that for any tree T of order $n \geq 2$ with ℓ leaves and s support vertices, $\gamma_R^L(T) \geq (2n + (\ell - s) + 2)/3$, and characterize all trees that achieve equality for this bound. In Section 3, we show that for any tree T of order $n \geq 2$, with l leaves and s support vertices, $\gamma_R^L(T) \leq (4n + l + s)/5$, and characterize all trees that achieve equality for this bound.

If $f = (V_0, V_1, V_2)$ is a $\gamma_R(G)$-function, then for any vertex $v \in V_2$, we define $pn(v, V_0) = \{u \in V_0 : N(u) \cap V_2 = \{v\}\}$. The following theorem was proved in [4].

Theorem 1 (Blidia *et al.* [4]). *For any tree T of order $n \geq 2$, $\gamma_L(T) \geq \lceil (n+1)/3 \rceil$.*

2. LOWER BOUND

We begin with the following lemma.

Lemma 2. *If T is a tree with ℓ leaves and s support vertices, and $f = (V_0, V_1, V_2)$ is a $\gamma_R^L(T)$-function, then $|V_1| \geq \ell - s$.*

Proof. For any support vertex x, $|L(x) \cap V_1| \geq \ell_x - 1$, thus $|V_1| \geq \sum_{x \in S}(\ell_x - 1) = \sum_{x \in S} \ell_x - \sum_{x \in S} 1 = \ell - s$. ■

Theorem 3. *For any tree T of order $n \geq 2$ with ℓ leaves and s support vertices, $\gamma_R^L(T) \geq (2n + (\ell - s) + 2)/3$.*

Proof. Let T be a tree of order n, and $f = (V_0, V_1, V_2)$ be a $\gamma_R^L(T)$-function. Let $T_1, T_2, \ldots, T_k$ be the components of $T[V_0 \cup V_2]$, and let $|V(T_i)| = n_i$ for $i = 1, 2, \ldots, k$. Let $D_i = V_2 \cap V(T_i)$ for $i = 1, 2, \ldots, k$. Clearly, D_i is a LDS for T_i, and so $\gamma_L(T_i) \leq |D_i|$, for $i = 1, 2, \ldots, k$. By Theorem 1, $|D_i| \geq \gamma_L(T_i) \geq (n_i + 1)/3$ for $i = 1, 2, \ldots, k$. Hence, $(n - |V_1| + k)/3 \leq \sum_{i=1}^k \gamma_L(T_i) \leq \sum_{i=1}^k |D_i| = |V_2|$. Now since $|V_1| \geq \ell - s$ by Lemma 2, we conclude that $\gamma_R^L(T) = |V_1| + 2|V_2| \geq |V_1| + (2(n - |V_1| + k))/3 \geq (2n + |V_1| + 2k)/3 \geq (2n + (\ell - s) + 2)/3$. ■

Corollary 4. *For any tree T of order $n \geqslant 2$, $\gamma_R^L(T) \geq (2n+2)/3$.*

We next aim to characterize trees achieving equality in the bound of Theorem 3. For this purpose for each integer $r \geq 0$, we construct a family $\mathcal{T}_r$ of trees as follows.

- Let $\mathcal{T}_0$ be the collection of trees T that can be obtained from a sequence $T_1, T_2, \ldots, T_k = T$ ($k \geq 1$) of trees, where $T_1 = P_5$, and T_{i+1} can be obtained recursively from T_i by the following operation for $1 \leq i \leq k-1$.

Operation $\mathcal{O}_1$. Join a support vertex of T_i to a leaf of a path P_3.

- For $r \geq 1$, let $\mathcal{T}_r$ be the class of trees T that can be obtained from a tree $T_0 \in \mathcal{T}_0$ by adding r leaves to at most r support vertices of T_0.

The following lemma plays a key role for the next section.

Lemma 5. *Let T be a tree of order $n \geq 3$ with $\gamma_R^L(T) = (2n+2)/3$. Then*
(1) $|V_1| = 0$ *for every $\gamma_R^L(T)$-function $f = (V_0, V_1, V_2)$.*
(2) *T has no strong support vertex.*
(3) *If $P = x_0 - x_1 - \cdots - x_d$ is a diametrical path of T, then* $\deg(x_{d-1}) = \deg(x_{d-2}) = 2$*, and x_{d-3} is a support vertex.*
(4) *If $P = x_0 - x_1 - \cdots - x_d$ is a diametrical path of T, and $T' = T - \{x_d, x_{d-1}, x_{d-2}\}$, then $\gamma_R^L(T') = (2|V(T')|+2)/3$.*

Proof. (1) Suppose that $f = (V_0, V_1, V_2)$ is a $\gamma_R^L(T)$-function such that $|V_1| > 0$. Let $v \in V_1$. If v is a leaf then by Corollary 4, we have $\frac{2n}{3} \leq \gamma_R^L(T - v) \leq w(f) - 1 = (2n-1)/3$, a contradiction. Thus v is not a leaf. Let $T_1, T_2, \ldots, T_k$ ($k \geq 2$) be the components of $T - \{v\}$, and $|V(T_i)| = n_i$ for $i = 1, \ldots, k$. For $i = 1, \ldots, k$, since $f|_{V(T_i)}$ is an LRDF for T_i, by Corollary 4 we obtain that $\frac{2n+2}{3} \leq \sum_{i=1}^{k} \frac{2n_i+2}{3} \leq \sum_{i=1}^{k} \gamma_R^L(T_i) \leq w(f) - 1 = (2n-1)/3$, a contradiction.

(2) The result follows from Lemma 2 and part (1).

(3) By part (2), $\deg(x_{d-1}) = 2$. Let $f = (V_0, V_1, V_2)$ be a $\gamma_R^L(T)$-function. Moreover, by parts (1) and (2) we may assume that $f(u) = 0$ for any leaf u, and $f(u) = 2$ for any support vertex u. Assume that $\deg(x_{d-2}) \geq 3$. If x_{d-2} is a support vertex then replacing $f(x_d)$ and $f(x_{d-1})$ by 1 yields a $\gamma_R^L(T)$-function, a contradiction to part (1). Thus x_{d-2} is not a support vertex. Then any vertex of $N(x_{d-2}) - \{x_{d-3}\}$ is a support vertex of degree two. If $\deg(x_{d-2}) \geq 4$ then replacing $f(x_d)$ and $f(x_{d-1})$ by 1 yields an LRDF for T, a contradiction to part (1). Assume that $\deg(x_{d-2}) = 3$. Observe that $f(x_{d-2}) = 0$. Let T' be the component of $T - x_{d-2}x_{d-3}$ that contains x_{d-3}. By Corollary 4, $\gamma_R^L(T') \geq (2(n-5)+2)/3$. But $f|_{V(T')}$ is an LRDF for T', and thus $(2(n-5)+2)/3 \leq \gamma_R^L(T') \leq w(f|_{V(T')}) = \gamma_R^L(T) - 4 = (2n+2)/3 - 4$, a contradiction. Thus $\deg(x_{d-2}) = 2$. Since $f(x_{d-1}) = 2$, from part (1) we obtain that $f(x_{d-2}) = 0$, and thus $f(x_{d-3}) = 2$.

Suppose now that x_{d-3} is not a support vertex. Assume that $\deg(x_{d-3}) = 2$. Clearly, we may assume that $f(x_{d-4}) = 0$, since otherwise replacing $f(x_d)$ and $f(x_{d-1})$ by 1 yields an $\gamma_R^L(T)$-function, a contradiction. By the same reason, we obtain that $N(x_{d-4}) \cap V_2 = \{x_{d-3}\}$. So x_{d-4} is neither a support vertex nor adjacent to a support vertex. Let $T_0, T_1, T_2, \ldots, T_l$ be the components of $T - x_{d-4}$, where T_0 contains x_{d-3}. Clearly, $f|_{V(T_i)}$ is an LRDF for T_i, and by Corollary 4, $w(f|_{V(T_i)}) \geq \gamma_R^L(T_i) \geq (2|V(T_i)| + 2)/3$ for $i = 1, 2, \ldots, l$. Thus

$$\begin{aligned}(2n-8)/3 &\leq (2(n-5)+2l)/3 = \sum_{i=1}^{l}(2|V(T_i)|+2)/3 \leq \sum_{i=1}^{l}\gamma_R^L(T_i)\\ &\leq \sum_{i=1}^{l} w(f|_{V(T_i)}) = w(f) - 4 = (2n+2)/3 - 4 = (2n-10)/3,\end{aligned}$$

a contradiction. Thus $\deg(x_{d-3}) \geq 3$. Let a_1 be a leaf of T such that the $d(x_{d-3}, a_1)$ is minimum and the shortest path from a_1 to x_{d-3} does not intersect P. Clearly, $d(x_{d-3}, a_1) \in \{2, 3\}$. Assume that $d(x_{d-3}, a_1) = 2$. Let $b_1 \in N(a_1) \cap N(x_{d-3})$. Thus $\deg(b_1) = 2$ by part (2). Then $f(b_1) = 2$, and so replacing $f(a_1)$ and $f(b_1)$ by 1 yields a $\gamma_R^L(T)$-function, a contradiction. Thus $d(x_{d-3}, a) = 3$. Therefore, any vertex of $N(x_{d-3}) - \{x_{d-4}\}$ has degree two and is adjacent to a support vertex of degree two. Let $N(x_{d-3}) - \{x_{d-4}, x_{d-2}\} = \{c_1, \ldots, c_k\}$, where $k = \deg(x_{d-3}) - 2$. Then c_i is adjacent to a support vertex b_i with $\deg(b_i) = 2$, for $i = 1, 2, \ldots, k$. Let a_i be the leaf adjacent to b_i for $i = 1, 2, \ldots, k$. Then $f(b_i) = 2$ and $f(a_i) = f(c_i) = 0$ for $i = 1, 2, \ldots, k$. Note that we may assume that $f(x_{d-4}) = 0$, since otherwise replacing $f(x_{d-1})$ and $f(x_d)$ by 1 yields a $\gamma_R^L(T)$-function, a contradiction. Thus x_{d-4} is neither a support vertex nor adjacent to a support vertex. By the same reason, $N(x_{d-4}) \cap V_2 = \{x_{d-3}\}$. Let $T_0, T_1, T_2, \ldots, T_l$ be the components of $T - x_{d-4}$, where T_0 contains x_{d-3}. Clearly, $f|_{V(T_i)}$ is an LRDF for T_i, and by Corollary 4, $w(f|_{V(T_i)}) \geq \gamma_R^L(T_i) \geq (2|V(T_i)| + 2)/3$ for $i = 1, 2, \ldots, l$. Thus

$$\begin{aligned}(2n-6k-8)/3 &\leq 2/3 + 2/3(n-3k-5) \leq 2/3 + 2/3\sum_{i=1}^{l}|V(T_i)|\\ &\leq \sum_{i=1}^{l}(2|V(T_i)|+2)/3 \leq \sum_{i=1}^{l} w(f|_{V(T_i)}) = w(f) - 2(k+1) - 2\\ &= (2n+2)/3 - 2k - 4 = (2n-6k-10)/3,\end{aligned}$$

a contradiction.

(4) By part (3), $\deg(x_{d-1}) = \deg(x_{d-2}) = 2$ and x_{d-3} is a support vertex. Let $f = (V_0, V_1, V_2)$ be a $\gamma_R^L(T)$-function. As seen earlier, $|V_1| = 0$, $f(x_d) = f(x_{d-2}) = 0$ and $f(x_{d-1}) = 2$. Therefore, $f|_{T'}$ is an LRDF for T'. By Corollary 4, $(2|V(T')|+2)/3 \leq \gamma_R^L(T') \leq w(f|_{T'}) = \gamma_R^L(T) - 2 = (2n+2)/3 - 2 = (2|V(T')| + 2)/3$. Therefore, $\gamma_R^L(T') = (2|V(T')| + 2)/3$. ■

We are now ready to characterize trees achieving equality in the bound of Theorem 3.

Theorem 6. *For a tree T of order $n \geq 2$ with ℓ leaves and s support vertices, $\gamma_R^L(T) = (2n + (\ell - s) + 2)/3$ if and only if $T = K_2$ or $T \in \mathcal{T}_k$ for some integer $k \geq 0$.*

Proof. Let $T \neq K_2$ be a tree of order n with ℓ leaves and s support vertices. We proceed with two claims.

Claim 1. *$\gamma_R^L(T) = (2n + 2)/3$ if and only if $T \in \mathcal{T}_0$.*

Proof. Assume that $\gamma_R^L(T) = (2n + 2)/3$. We show by induction on n that $T \in \mathcal{T}_0$. For the base step of the induction it is easy to see that P_5 is the smallest tree T for which $\gamma_R^L(T) = (2n+2)/3$. Assume that any tree T' of order $5 < n' < n$ and such that $\gamma_R^L(T') = (2n' + 2)/3$ belongs to $\mathcal{T}_0$. Let $P = x_0 - x_1 - \cdots - x_d$ be a diametrical path of T. By Lemma 5(3), $\deg(x_{d-1}) = \deg(x_{d-2}) = 2$, and x_{d-3} is a support vertex. Let $T_1 = T - \{x_d, x_{d-1}, x_{d-2}\}$. By Lemma 5(4), $\gamma_R^L(T_1) = (2|V(T_1)| + 2)/3$. By the inductive hypothesis, $T_1 \in \mathcal{T}_0$. Hence T is obtained from T_1 by Operation $\mathcal{O}_1$, and thus $T \in \mathcal{T}_0$. For the converse it is sufficient to show that if $\gamma_R^L(T_i) = (2|V(T_i)| + 2)/3$ and T_{i+1} is obtained from T_i by the operation $\mathcal{O}_1$, then $\gamma_R^L(T_{i+1}) = (2|V(T_{i+1})| + 2)/3$, and then the result follows by an induction on the number of operations performed to construct a tree $T \in \mathcal{T}_0$. Let $\gamma_R^L(T_i) = (2|V(T_i)| + 2)/3$, and T_{i+1} be obtained from T_i by joining a support vertex $v \in V(T_i)$ to the leaf x of a path $P_3 : xyz$. Let f be a $\gamma_R^L(T_i)$-function. By Lemma 5(1) and (2), we may assume that $f(v) = 2$. Then $g : V(T_{i+1}) \longrightarrow \{0, 1, 2\}$ defined by $g(x) = g(z) = 0, g(y) = 2$ and $g(u) = f(u)$ for any $u \in V(T_i)$, is an LRDF for T_{i+1}. By Corollary 4, $(2|V(T_{i+1})| + 2)/3 \leq \gamma_R^L(T_{i+1}) \leq w(g) = \gamma_R^L(T) + 2 = (2|V(T_i)| + 2)/3 + 2 = (2|V(T_{i+1}) + 2)/3$. Therefore, $\gamma_R^L(T_{i+1}) = (2|V(T_{i+1})| + 2)/3$. □

Claim 2. *$\gamma_R^L(T) = (2n + (\ell - s) + 2)/3$, with $\ell \neq s$, if and only if $T \in \mathcal{T}_k$ for some integer $k \geq 1$.*

Proof. Assume that $\gamma_R^L(T) = (2n+(\ell-s)+2)/3$, and $\ell \neq s$. Let $f = (V_0, V_1, V_2)$ be a $\gamma_R^L(T)$-function. For any support vertex x, $f(u) = 1$ for at least $\ell_x - 1$ leaves $u \in N(x)$ by Lemma 2. Let T' be a tree obtained from T by removing $\ell_x - 1$ leaves u of any strong support vertex x with $f(u) = 1$. Then $f|_{T'}$ is a LRDF for T', and so $\gamma_R^L(T') \leq \gamma_R^L(T) - (l - s) = (2(n - (l - s) + 2))/3 = (2|V(T')| + 2)/3$. Corollary 4 implies that $\gamma_R^L(T') = (2|V(T')| + 2)/3$. Now Claim 1 implies that $T' \in \mathcal{T}_0$, and so $T \in \mathcal{T}_k$, where $k = l - s$. Conversely, let $T \in \mathcal{T}_k$ for some integer $k \geq 1$. Thus T is obtained from a tree $T' \in \mathcal{T}_0$ by adding k leaves to at most k support vertices of T'. By Claim 1, $\gamma_R^L(T') = (2|V(T')| + 2)/3$. Let f' be a $\gamma_R^L(T')$-function. We extend f' to a LRDF for T by assigning 1 to any vertex of

$V(T) - V(T')$, and thus $\gamma_R^L(T) \le \gamma_R^L(T') + l - s = (2|V(T')| + 2)/3 + l - s = (2(|V(T')| + l - s) + l - s + 2)/3 = (2|V(T)| + (\ell - s) + 2)/3$. Now Theorem 3 implies that $\gamma_R^L(T) = (2n + (\ell - s) + 2)/3$. □

Now the proof follows by Claims 1 and 2. ■

3. Upper Bound

Lemma 7. *If T' is a tree and T is obtained from T' by joining a leaf of T' to a leaf of a path P_5, then $\gamma_R^L(T) = \gamma_R^L(T') + 4$.*

Proof. Let T be obtained from a tree T' by joining a leaf v of T' to the leaf a of a path $P_5 : abcde$. If $f = (V_0, V_1, V_2)$ is a $\gamma_R^L(T')$-function, then $g = (V_0 \cup \{a, c, e\}, V_1, V_2 \cup \{b, d\})$ is an LRDF for T, and so $\gamma_R^L(T) \le \gamma_R^L(T') + 4$. Let $h = (V_0, V_1, V_2)$ be a $\gamma_R^L(T)$-function. If $a \notin V_2$, then $h(a) + h(b) + h(c) + h(d) + h(e) = 4$ and $h|_{V(T')}$ is an LRDF for T', so $\gamma_R^L(T') \le \gamma_R^L(T) - 4$. If $a \in V_2$, then $h(a) + h(b) + h(c) + h(d) + h(e) = 5$, so $\gamma_R^L(T') \le w(h|_{V(T')}) + 1 = \gamma_R^L(T) - 4$. Thus $\gamma_R^L(T) = \gamma_R^L(T') + 4$. ■

Similarly the following is verified.

Lemma 8. *Let T' be a tree with a vertex w of degree at least two and $\gamma_R^L(T' - w) \ge \gamma_R^L(T')$. If T is obtained from T' by joining w to the center of a path P_9, then $\gamma_R^L(T) = \gamma_R^L(T') + 8$.*

Theorem 9. *For any tree T of order $n \ge 2$, with ℓ leaves and s support vertices, $\gamma_R^L(T) \le (4n + \ell + s)/5$.*

Proof. We use an induction on the order $n = n(T)$ of a tree T. The base step is obvious for $n \le 4$. Assume that for any tree T' of order $n' < n$, with ℓ' leaves and s' support vertices, $\gamma_R^L(T') \le (4n' + \ell' + s')/5$. Now consider the tree T of order $n \ge 5$, with ℓ leaves and s support vertices. Assume that T has a strong support vertex v, and u is a leaf adjacent to v. Let $T' = T - u$. Clearly, $\gamma_R^L(T) \le \gamma_R^L(T') + 1$. By the induction hypothesis, $\gamma_R^L(T) \le \gamma_R^L(T') + 1 \le (4n' + \ell' + s')/5 + 1 = (4(n-1) + (l-1) + s)/5 + 1 = (4n + l + s)/5$. Next assume that T has an edge $e = uv$ with $\deg(u) \ge 3$ and $\deg(v) \ge 3$. Let T_1 and T_2 be the components of $T - e$, with $u \in V(T_1)$ and $v \in V(T_2)$. Assume that T_i has order n_i, ℓ_i leaves and s_i support vertices, for $i = 1, 2$. By the induction hypothesis, $\gamma_R^L(T) \le \gamma_R^L(T_1) + \gamma_R^L(T_2) \le (4n_1 + \ell_1 + s_1)/5 + (4n_2 + \ell_2 + s_2)/5 = (4n + \ell + s)/5$. Thus the following claims hold.

Claim 1. *T has no strong support vertex.*

Claim 2. *For each edge $e = uv$, $\deg(u) \le 2$ or $\deg(v) \le 2$.*

We root T at a leaf x_0 of a diametrical path $x_0x_1\cdots x_d$ from x_0 to a leaf x_d farthest from x_0. By Claim 1, $d\geq 3$. If $d=3$ then T is a double-star, and it can be easily seen that $\gamma_R^L(T)=(4n+\ell+s)/5$. Thus assume that $d\geq 4$.

By Claim 1, $\deg(x_{d-1})=2$. Assume that $\deg(x_{d-2})\geq 3$. Assume that x_{d-2} is a support vertex. Let u be the unique leaf adjacent to x_{d-2}. Let $T'=T-u$. By the inductive hypothesis, $\gamma_R^L(T)\leq\gamma_R^L(T')+1\leq(4(n-1)+(\ell-1)+(s-1))/5+1<(4n+\ell+s)/5$. Thus assume that x_{d-2} is not a support vertex. Let u be a child of x_{d-2} different from x_{d-1}. By Claim 1, $\deg(u)=2$. Let v be the child of u, and $T'=T-\{u,v\}$. By the inductive hypothesis, $\gamma_R^L(T)\leq\gamma_R^L(T')+2\leq(4(n-2)+(\ell-1)+s-1)/5+2=(4n+\ell+s)/5$. We thus assume that $\deg(x_{d-2})=2$.

Assume that $\deg(x_{d-3})\geq 3$. Assume that x_{d-3} is a support vertex. Let u be the unique leaf adjacent to x_{d-3}. Let $T'=T-u$. By the inductive hypothesis, $\gamma_R^L(T)\leq\gamma_R^L(T')+1\leq(4(n-1)+(\ell-1)+s-1)/5+1<(4n+\ell+s)/5$. Thus assume that x_{d-3} is not a support vertex. Let u be a child of x_{d-3} different from x_{d-2}. Assume that u is a support vertex. By Claim 1, $\deg(u)=2$. Let v be the child of u. Let $T'=T-\{u,v\}$. By the inductive hypothesis, $\gamma_R^L(T)\leq\gamma_R^L(T')+2\leq(4(n-2)+(\ell-1)+s-1)/5+2=(4n+\ell+s)/5$. Thus assume that u is not a support vertex. Thus any child of u is a support vertex of degree two by Claim 1. Furthermore, since $\deg(x_{d-3})\geq 3$, we deduce that $d\geq 6$, and this implies that $x_{d-5}\neq x_0$. Let $\deg(x_{d-3})=k+1$. By Claim 2, $\deg(x_{d-4})=2$. Let $T'=T-T_{x_{d-4}}$. Assume that T' has n' vertices, ℓ' leaves and s' support vertices. By the inductive hypothesis, $\gamma_R^L(T')\leq(4n'+\ell'+s')/5$. But $\ell'\leq\ell-k+1$, $s'\leq s-k+1$, and $n'=n-3k-2$. Let f be a $\gamma_R^L(T')$-function. We extend f to an LRDF for T by assigning 2 to x_{d-3} and any vertex of $T_{x_{d-4}}$ at distance two from x_{d-3}, and 0 to any other vertex of $T_{x_{d-4}}$. Thus $\gamma_R^L(T)\leq\gamma_R^L(T')+2k+2\leq(4n'+\ell'+s')/5+2k+2\leq(4n+\ell+s-4k+4)/5\leq(4n+\ell+s)/5$. Thus assume that $\deg(x_{d-3})=2$.

Assume that $\deg(x_{d-4})\geq 3$. As before, we can assume that x_{d-4} is not a support vertex, and is not adjacent to a support vertex of degree two. By Claim 2, $\deg(x_{d-5})=2$, and also any child of x_{d-4} has degree two. If there is a leaf $u\neq x_d$ of $T_{x_{d-5}}$ at distance four from x_{d-4} then any internal vertex in the path from u to x_{d-4} has degree two, since u plays the same role of x_d. Thus any leaf u of $T_{x_{d-4}}$ is at distance 3 or 4 from X_{d-4}, and any internal vertex in the path from u to x_{d-4} has degree two. Let k_1 be the number of leaves of $T_{x_{d-5}}$ at distance four from x_{d-4}, and k_2 be the number of leaves of $T_{x_{d-5}}$ at distance three from x_{d-4}. Then $\deg(x_{d-4})=k_1+k_2+1$. Since $\deg(x_{d-4})\geq 3$, we obtain that $d\geq 7$, and this implies that $x_{d-6}\neq x_0$. Let $T'=T-T_{x_{d-5}}$. Assume that T' has n' vertices, ℓ' leaves and s' support vertices. By the inductive hypothesis, $\gamma_R^L(T')\leq(4n'+\ell'+s')/5$. But $\ell'\leq\ell-k_1-k_2+1$, $s'\leq s-k_1-k_2+1$, and $n'=n-4k_1-3k_2-2$. Let f be a $\gamma_R^L(T')$-function. We extend f to an LRDF for T by assigning 2 to x_{d-4} and any vertex of $T_{x_{d-5}}$ at distance two from x_{d-4}, 1 to any vertex of $T_{x_{d-5}}$ at distance four from x_{d-4}, and 0 to any other vertex of $T_{x_{d-5}}$. Thus $\gamma_R^L(T) \leq \gamma_R^L(T') + 3k_1 + 2k_2 + 2 \leq (4n' + \ell' + s')/5 + 3k_1 + 2k_2 + 2 \leq (4n + \ell + s - 3k_1 - 4k_2 + 4)/5 < (4n + \ell + s)/5$.

Thus assume that $\deg(x_{d-4}) = 2$. Let $T' = T - T_{x_{d-5}}$. Assume that T' has n' vertices, ℓ' leaves and s' support vertices. By the inductive hypothesis, $\gamma_R^L(T') \leq (4n' + \ell' + s')/5$. But $\ell' \leq \ell$, $s' \leq s$, and $n' = n - 5$. Let f be a $\gamma_R^L(T')$-function. We extend f to an LRDF for T by assigning 2 to x_{d-3} and x_{d-1}, and 0 to x_{d-4}, x_{d-2} and x_d. Thus $\gamma_R^L(T) \leq \gamma_R^L(T') + 4 \leq (4n' + \ell' + s')/5 + 4 \leq (4n + \ell + s)/5$. ■

We next aim to characterize trees achieving equality for the bound of Theorem 3. A vertex w of degree at least two in a tree T is called a *special vertex* if the following conditions hold:

(1) If $f(w) = 2$ for a $\gamma_R^L(T)$-function $h = (V_0, V_1, V_2)$, then $pn(w, V_0) \neq \emptyset$.

(2) If $f(w) = 1$ for a $\gamma_R^L(T)$-function $h = (V_0, V_1, V_2)$, then $N(w) \cap V_2 = \emptyset$.

Let $\mathcal{T}$ be the collection of trees T that can be obtained from a sequence $T_1, T_2, \ldots, T_k = T$ $(k \geq 1)$ of trees, where $T_1 = P_4$, and T_{i+1} can be obtained recursively from T_i by one of the following operations for $1 \leq i \leq k - 1$.

Operation $\mathcal{O}_1$. Assume that w is a support vertex of T_i. Then T_{i+1} is obtained from T_i by adding a leaf to w.

Operation $\mathcal{O}_2$. Assume that w is a leaf of T_i. Then T_{i+1} is obtained from T_i by joining w to a leaf of a path P_5.

Operation $\mathcal{O}_3$. Assume that w is a specialvertex of T_i. Then T_{i+1} is obtained from T_i by joining w to a leaf of a path P_2.

Operation $\mathcal{O}_4$. Assume that w is a vertex of T_i of degree at least two and $\gamma_R^L(T_i - w) \geq \gamma_R^L(T_i)$. Then T_{i+1} is obtained from T_i by joining w to the center of a path P_9.

Lemma 10. *If $\gamma_R^L(T_i) = (4n(T_i) + \ell(T_i) + s(T_i))/5$, and T_{i+1} is obtained from T_i by Operation $\mathcal{O}_j$, for $j = 1, 2, 3, 4$, then $\gamma_R^L(T_{i+1}) = (4n(T_{i+1}) + \ell(T_{i+1}) + s(T_{i+1}))/5$.*

Proof. Let $\gamma_R^L(T_i) = (4n_i + \ell_i - 2 + s_i)/5$, where $n_i = n(T_i)$, $\ell_i = \ell(T_i)$ and $s_i = s(T_i)$. Assume that T_{i+1} is obtained from T_i by Operation $\mathcal{O}_1$. Let T_{i+1} be obtained from T_i by adding a leaf v to a support vertex w of T_i. Then $\gamma_R^L(T_{i+1}) \leq \gamma_R^L(T_i) + 1$. Let $f = (V_0, V_1, V_2)$ be a $\gamma_R^L(T_{i+1})$-function, without loss of generality, we may assume that $v \in V_1$. Then $f = (V_0, V_1 - \{v\}, V_2)$ is an LRDF for T_i, implying that $\gamma_R^L(T_i) \leq \gamma_R^L(T_{i+1}) - 1$. Thus $\gamma_R^L(T_{i+1}) = \gamma_R^T(T_i) + 1$. Now $\gamma_R^L(T_{i+1}) = (4n(T_i) + \ell(T_i) + s(T_i))/5 + 1 = (4(n(T_i) + 1) + (\ell(T_i) + 1) + s(T_i))/5 = (4n(T_{i+1}) + \ell(T_{i+1}) + s(T_{i+1}))/5$.

Next assume that T_{i+1} is obtained from T_i by Operation $\mathcal{O}_2$. By Lemma 7, $\gamma_R^L(T_{i+1}) = \gamma_R^L(T_i) + 4$. Now $\gamma_R^L(T_{i+1}) = (4n(T_i) + \ell(T_i) + s(T_i))/5 + 4 = (4(n(T_i) + 5) + \ell(T_i) + s(T_i))/5 = (4n(T_{i+1}) + \ell(T_{i+1}) + s(T_{i+1}))/5$.

Now assume that T_{i+1} is obtained from T_i by Operation $\mathcal{O}_3$. Let T_{i+1} be obtained from T_i by joining a special vertex v of T_i to the leaf a of a path $P_2 : ab$. Suppose that $\gamma_R^L(T_{i+1}) = \gamma_R^L(T_i) + 1$. Let h be a $\gamma_R^L(T_{i+1})$-function. Assume that $h(a) = 2$. Clearly, we may assume that $h(b) = 0$. If $h(v) \neq 0$, then $h|_{V(T_i)}$ is an LRDF for T_i of weight less than $\gamma_R^L(T_i)$, a contradiction. Thus $h(v) = 0$. Since h is an LRDF for T_{i+1}, there is a vertex $w \in N(v) - \{a\}$ such that $h(w) = 2$. Now h' defined on $V(T_i)$ by $h'(v) = 1$ and $h'(x) = h(x)$ otherwise, is an LRDF for T_i. Clearly, h' is a $\gamma_R^L(T_i)$-function. This is a contradiction, since v is a special vertex of T_i. If $h(a) = 1$, then $h(b) = 1$ and we can replace $h(a)$ by 2 and $h(b)$ by 0, and as before, get a contradiction. Thus $h(a) = 0$. If $h(b) = 2$, then we replace $h(a)$ by 2 and $h(b)$ by 0, and as before, get a contradiction. Thus $h(b) = 1$, and so $h(v) = 2$. Thus $h|_{V(T_i)} = (V_0, V_1, V_2)$ is a $\gamma_R^L(T_i)$-function with $pn(v, V_0) = \emptyset$. This is a contradiction, since v is a special vertex of T_i. Thus $\gamma_R^L(T_{i+1}) = \gamma_R^L(T_i)+2$. Now $\gamma_R^L(T_{i+1}) = (4(n(T_i)+2)+(\ell(T_i)+1)+(s(T_i)+1))/5 = (4n(T_{i+1}) + \ell(T_{i+1}) + s(T_{i+1}))/5$.

Finally assume that T_{i+1} is obtained from T_i by Operation $\mathcal{O}_4$. By Lemma 8, $\gamma_R^L(T_{i+1}) = \gamma_R^L(T_i)+8$. Now $\gamma_R^L(T_{i+1}) = (4n(T_i)+\ell(T_i)+s(T_i))/5+8 = (4(n(T_i)+9) + (\ell(T_i) + 2) + (s(T_i) + 2))/5 = (4n(T_{i+1}) + \ell(T_{i+1}) + s(T_{i+1}))/5$. ■

By a simple induction on the operations performed to construct a tree $T \in \mathcal{T}$ and Lemma 10 we obtain the following.

Lemma 11. *For any tree $T \in \mathcal{T}$ of order $n \geq 2$ with ℓ leaves and s support vertices, $\gamma_R^L(T) = (4n + \ell + s)/5$.*

Theorem 12. *For a tree T of order $n \geq 2$ with ℓ leaves and s support vertex, $\gamma_R^L(T) = (4n + \ell + s)/5$ if and only if $T = K_{1,n-1}$ or $T \in \mathcal{T}$.*

Proof. We use an induction on the order n of a tree $T \neq K_{1,n-1}$ with ℓ leaves, s support vertices and $\gamma_R^L(T) = (4n + \ell + s)/5$ to show that $T \in \mathcal{T}$. Since $T \neq K_{1,n-1}$, for the basic step consider a path P_4, and note that $P_4 \in \mathcal{T}$. Assume that any tree T of order $n' < n$, with ℓ' leaves, s' support vertices and $\gamma_L(T') = (4n' + \ell' + s')/5$ belongs to $\mathcal{T}$. Let $n = n(T) \geq 5$.

Assume that T has a support vertex u with $\deg(u) \geq 3$. Let v be a leaf adjacent to u, and $T' = T - v$. We can easily see that $\gamma_R^L(T) = \gamma_R^L(T') + 1$. If u is not a strong support vertex, then $\gamma_R^L(T) \leq \gamma_R^L(T') + 1 = (4(n-1) + \ell - 1 + s - 1)/5 + 1 < (4n + \ell + s)/5$, a contradiction. Thus u is a strong support vertex. Then $\gamma_R^L(T') = \gamma_R^L(T) - 1 = (4n + \ell + s)/5 - 1 = (4(n-1) + (\ell - 1) + s)/5 = (4n(T') + \ell(T') + s(T'))/5$. By the inductive hypothesis, $T' \in \mathcal{T}$. Hence T is obtained from T' by Operation $\mathcal{O}_1$. Thus we assume that the following claim holds.

Claim 1. *Any support vertex of T is of degree two.*

We root T at a leaf x_0 of a diametrical path $x_0 x_1 \cdots x_d$ from x_0 to a leaf x_d farthest from x_0. Clearly, $d \geq 3$. Since $n > 4$ and T has no strong support vertex, we find that $d \geq 4$. Clearly, $\deg(x_1) = \deg(x_{d-1}) = 2$. Assume that $d = 4$. If $\deg(x_2) = 2$ then $T = P_5$, and $\gamma_R^L(T) = 4 < (4n + \ell + s)/5$, a contradiction. Thus $\deg(x_2) > 2$. By Claim 1, x_2 is not a support vertex. Then T has $\deg(x_2)$ support vertices of degree two, and we can see that $(L(T) \cup \{x_2\}, \emptyset, S(T))$ is an LRDF for T, implying that $\gamma_R^L(T) \leq 2s < (4n + \ell + s)/5$, since $n = 2s + 1$ and $\ell = s$. This is a contradiction. Thus $d \geq 5$.

We show that $\deg(x_{d-2}) = 2$. Assume that $3 \leq \deg(x_{d-2}) = k+1$. By Claim 1, x_{d-2} is not a support vertex. Thus any child of x_{d-2} is a support vertex of degree two. Let $T' = T - T_{x_{d-2}}$, and $f = (V_0, V_1, V_2)$ be a $\gamma_R^L(T')$-function. Then $h = (V_0 \cup S(T_{x_{d-2}}) \cup \{x_{d-2}\}, V_1, V_2 \cup S(T_{x_{d-2}}))$ is an LRDF for T, implying by Theorem 9 that $\gamma_R^L(T) \leq \gamma_R^L(T') + 2k \leq (4(n - 2k - 1) + (\ell - k + 1) + (s - k + 1))/5 + 2k < (4n + \ell + s)/5$, a contradiction. Thus $\deg(x_{d-2}) = 2$.

We next show that $\deg(x_{d-3}) = 2$. Suppose that $\deg(x_{d-3}) \geq 3$. By Claim 1, x_{d-3} is not a support vertex. If there is a leaf v of $T_{x_{d-3}}$ different from x_d at distance three from x_{d-3}, then any internal vertex in the path from v to x_{d-3} is of degree two, since v plays the same role of x_d. Then any child of x_{d-3} is a support vertex of degree two or is a vertex of degree two and adjacent to a support vertex of degree two. Let k_1 be the number of leaves of $T_{x_{d-3}}$ at distance three from x_{d-3} and k_2 be the number of leaves of $T_{x_{d-3}}$ at distance two from x_{d-3}. Note that $\deg(x_{d-3}) = k_1 + k_2 + 1$. Assume that $deg(x_{d-4}) \geq 3$. Let $T' = T - T_{x_{d-3}}$, and let $f = (V_0, V_1, V_2)$ be a $\gamma_R^L(T')$-function. If $k_2 = 0$ then $h = (V_0 \cup V(T_{x_{d-2}}) - (S(T_{x_{d-3}}) \cup \{x_{d-3}\}), V_1, V_2 \cup S(T_{x_{d-3}}) \cup \{x_{d-3}\})$ is an LRDF for T, implying by Theorem 9 that $\gamma_R^L(T) \leq \gamma_R^L(T') + 2k_1 + 2 < (4n + \ell + s)/5$, a contradiction. Thus assume that $k_2 > 0$. Let u be a leaf at distance two from x_{d-3} and v be the father of u. Then $h = (V_0 \cup V(T_{x_{d-2}}) - (S(T_{x_{d-3}} - \{v\}) \cup \{x_{d-3}, u\}), V_1 \cup \{u\}, V_2 \cup S(T_{x_{d-3}} - \{v\}) \cup \{x_{d-3}\})$ is an LRDF for T, implying by Theorem 9 that $\gamma_R^L(T) \leq \gamma_R^L(T') + 2k_1 + 2k_2 + 1 < (4n + \ell + s)/5$, a contradiction. We deduce that $\deg(x_{d-4}) = 2$. Assume that $k_2 = 0$. Let $T' = T - T_{x_{d-4}}$, and let $f = (V_0, V_1, V_2)$ be a $\gamma_R^L(T')$-function. Then $h = (V_0 \cup V(T_{x_{d-2}}) - S(T_{x_{d-4}}), V_1, V_2 \cup S(T_{x_{d-4}}))$ is an LRDF for T, implying by Theorem 9 that $\gamma_R^L(T) \leq \gamma_R^L(T') + 2k_1 + 2k_2 + 2 < (4n + \ell + s)/5$, a contradiction. Thus assume that $k_2 > 0$. Let $T' = T - T_{x_{d-3}}$, and let $f = (V_0, V_1, V_2)$ be a $\gamma_R^L(T')$-function. Let u be a leaf at distance two from x_{d-3} and v be the father of u. Then $h = (V_0 \cup V(T_{x_{d-2}}) - (S(T_{x_{d-3}} - \{v\}) \cup \{x_{d-3}, u\}), V_1 \cup \{u\}, V_2 \cup S(T_{x_{d-3}} - \{v\}) \cup \{x_{d-3}\})$ is an LRDF for T, implying by Theorem 9 that $\gamma_R^L(T) \leq \gamma_R^L(T') + 2k_1 + 2k_2 + 1 < (4n + \ell + s)/5$, a contradiction. We conclude that $\deg(x_{d-3}) = 2$.

Assume that $\deg(x_{d-4}) = 2$. If $\deg(x_{d-5}) \geq 3$, then let $T' = T - T_{x_{d-4}}$ and $f = (V_0, V_1, V_2)$ be a $\gamma_R^L(T')$-function. Then $h = (V_0 \cup \{x_d, x_{d-2}, x_{d-4}\}, V_1, V_2 \cup \{x_{d-1}, x_{d-3}\})$ is a LRDF function for T. Hence $\gamma_R^L(T) \leq \gamma_R^L(T') + 4 < (4n + \ell + s)/5$, a contradiction. Thus $\deg(x_{d-5}) = 2$. Since $\gamma_R^L(P_7) = 6 < (4(7) + 2 + 2)/5$, we find that $\deg(x_{d-6}) \geq 2$. Since $\gamma_R^L(P_8) = 7 < (4(8) + 2 + 2)/5$, we find that $\deg(x_{d-7}) \geq 2$. Thus x_{d-6} is not a support vertex. By Lemma 7, $\gamma_R^L(T') = \gamma_R^L(T)-4 = (4n+\ell+s)/5-4 = (4(n-5)+\ell+s)/5 = (4n(T')+\ell(T')+s(T'))/5$. By the inductive hypothesis, $T' \in \mathcal{T}$. Now T is obtained from T' by Operation $\mathcal{O}_2$.

Next assume that $\deg(x_{d-4}) \geq 3$. By Claim 1, x_{d-4} is not a support vertex. Suppose that there is a leaf v of $T_{x_{d-4}}$ at distance two from x_{d-4}. Let u be the father of v. Clearly, $\deg(u) = 2$. Let $T' = T - \{u, v\}$. Suppose that there is a $\gamma_R^L(T')$ function $f = (V_0, V_1, V_2)$ with $f(x_{d-4}) = 2$ and $pn(x_{d-4}, V_0) = \emptyset$. Then $(V_0 \cup \{u\}, V_1 \cup \{v\}, V_2)$ is an LRDF for T, and so $\gamma_R^L(T) \leq \gamma_R^L(T') + 1 < (4n + \ell + s)/5$, a contradiction. Thus there is no $\gamma_R^L(T')$ function $f = (V_0, V_1, V_2)$ with $f(x_{d-4}) = 2$ and $pn(x_{d-4}, V_0) = \emptyset$. Suppose that there is a $\gamma_R^L(T')$ function $f = (V_0, V_1, V_2)$ with $f(x_{d-4}) = 1$, and $N(x_{d-4}) \cap V_2 \neq \emptyset$. Then $(V_0 \cup \{v, x_{d-4}\}, V_1 - \{x_{d-4}\}, V_2 \cup \{u\})$ is an LRDF for T, and so $\gamma_R^L(T) \leq \gamma_R^L(T') + 1 < (4n + \ell + s)/5$, a contradiction. Thus x_{d-4} is a special vertex of T'. Clearly, $\gamma_R^L(T') + 1 \leq \gamma_R^L(T) \leq \gamma_R^L(T') + 2$. Suppose that $\gamma_R^L(T) = \gamma_R^L(T') + 1$. Let h be a $\gamma_R^L(T)$-function. Assume that $h(u) = 2$. Clearly, we may assume that $h(v) = 0$. If $h(x_{d-4}) \neq 0$, then $h|_{V(T')}$ is an LRDF for T' of weight less than $\gamma_R^L(T')$, a contradiction. Thus $h(x_{d-4}) = 0$. Since h is an LRDF for T, there is a vertex $w \in N(x_{d-4}) - \{u\}$ such that $h(w) = 2$. Now h' defined on $V(T')$ by $h'(x_{d-4}) = 1$ and $h'(x) = h(x)$ otherwise, is an LRDF for T'. Clearly, that h' is a $\gamma_R^L(T')$-function. This is a contradiction, since x_{d-4} is a special vertex of T'. If $h(u) = 1$, then $h(v) = 1$ and we can replace $h(u)$ by 2 and $h(v)$ by 0, and as before, get a contradiction. Thus $h(u) = 0$. If $h(v) = 2$, then we replace $h(u)$ by 2 and $h(v)$ by 0, and as before, get a contradiction. Thus $h(v) = 1$, and so $h(x_{d-4}) = 2$. Thus $h|_{V(T')} = (V_0, V_1, V_2)$ is a $\gamma_R^L(T')$-function with $pn(x_{d-4}, V_0) = \emptyset$. This is a contradiction, since x_{d-4} is a special vertex of T'. Thus $\gamma_R^L(T) = \gamma_R^L(T') + 2$. Now $\gamma_R^L(T') = \gamma_R^L(T) - 2 = (4n + \ell + s)/5 - 2 = (4(n - 2) + \ell - 1 + s - 1)/5 = (4n(T') + \ell(T') + s(T'))/5$. By the inductive hypothesis, $T' \in \mathcal{T}$. Thus T is obtained from T' by Operation $\mathcal{O}_3$.

Now, we assume that any leaf of $T_{x_{d-4}}$ has distance three or four from x_{d-4}. If there is a leaf v of $T_{x_{d-4}}$ at distance four from x_{d-4}, then any internal vertex in the path from v to x_{d-4} is of degree two, since v plays the role of x_d. Moreover, by Claim 1, if v is a leaf of $T_{x_{d-4}}$ at distance three from x_{d-4}, then any internal vertex in the path from v to x_{d-4} is of degree two. Let k_1 be the number of leaves of $T_{x_{d-4}}$ at distance four from x_{d-4} and k_2 be the number of leaves of $T_{x_{d-3}}$ at distance three from x_{d-4}. Note that $\deg(x_{d-4}) = k_1 + k_2 + 1$. Suppose that $\deg(x_{d-5}) = 2$. Let $T' = T - T_{x_{d-5}}$ and $f = (V_0, V_1, V_2)$ be a $\gamma_R^L(T')$-function. Then $h = (V_0 \cup W, V_1 \cup Z, V_2 \cup U)$ is a LRDF for T, where W is the set of vertices of $T_{x_{d-4}}$ at distance one or three of x_{d-4}, Z is the set of vertices at distance four from x_{d-4}, and U contains x_{d-4} and all vertices at distance two from x_{d-4}. Hence $\gamma_R^L(T) \leq \gamma_R^L(T') + 3k_1 + 2k_2 + 2 < (4n + \ell + s)/5$, a contradiction. Thus $\deg(x_{d-5}) \geq 3$. Let $T' = T - T_{x_{d-4}}$ and $f = (V_0, V_1, V_2)$ be a $\gamma_R^L(T')$-function.

Then $h = (V_0 \cup W, V_1 \cup Z, V_2 \cup U)$ is a $\gamma_R^L(T)$-function, where W is the set of vertices at distance one or three of x_{d-4}, Z is the set of vertices at distance four from x_{d-4}, and U contains x_{d-4} and vertices at distance two from x_{d-4}. Hence $\gamma_R^L(T) \leq \gamma_R^L(T') + 3k_1 + 2k_2 + 2$. If $k_2 \neq 0$ or $k_1 \geq 3$, then $\gamma_R^L(T) \leq \gamma_R^L(T') + 3k_1 + 2k_2 + 2 < (4n + \ell + s)/5$, a contradiction. Thus $k_2 = 0$ and $k_1 = 2$. By Lemma 8, $\gamma_R^L(T) = \gamma_R^L(T') + 8$. Thus $\gamma_R^L(T') = \gamma_R^L(T) - 8 = (4n + \ell + s)/5 - 8 = (4(n-9) + (\ell - 2) + (s-2))/5 = (4n(T') + \ell(T') + s(T'))/5$. By the inductive hypothesis, $T' \in \mathcal{T}$. Suppose $\gamma_R^L(T' - x_{d-5}) < \gamma_R^L(T')$. Let g be a $\gamma_R^L(T' - x_{d-5})$-function. We extend g to an LRDF for T by assigning 0 to x_{d-5} and the vertices of $T_{x_{d-4}}$ at distance one or three from x_{d-4}, 2 to x_{d-4} and the vertices of $T_{x_{d-4}}$ at distance two. Thus $\gamma_R^L(T) \leq \gamma_R^L(T' - x_{d-5}) + 8 < \gamma_R^L(T') + 8 < (4n + \ell + s)/5$, a contradiction. Hence $\gamma_R^L(T' - x_{d-5}) \geq \gamma_R^L(T')$. Now T is obtained from T' by Operation $\mathcal{O}_4$. The converse follows by Lemma 11. ∎

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