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Note

A note on the global offensive alliances in graphs

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ABSTRACT

A subset S of vertices in a graph $G = (V, E)$ is a global offensive alliance if for every vertex v not in S, at least half of the vertices in the closed neighborhood of v are in S. The global offensive alliance number of G is the minimum cardinality among all global offensive alliances in G. A global offensive alliance D is called a global strong offensive alliance if for every vertex v not in S, more than half of the vertices in the closed neighborhood of v are in S. The global strong offensive alliance number of G is the minimum cardinality among all global strong offensive alliances in G. In this paper, we present new upper bounds for the global offensive alliance number as well as the global strong offensive alliance number of a graph. We improve previous upper bounds given in Harutyunyan (2014).

1. Introduction

For graph theory notation and terminology not given here we refer to [11], and for the probabilistic methods notation and terminology we refer to [1]. We consider finite and simple graphs G with vertex set $V = V(G)$ and edge set $E(G)$. The number of vertices of G is called the *order* of G and is denoted by $n = n(G)$. The *open neighborhood* of a vertex $v \in V$ is $N(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood* of v is $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex v, denoted by $\deg(v)$ (or $\deg_G(v)$ to refer to G), is the cardinality of its open neighborhood. We denote by $\delta(G)$ and $\Delta(G)$, the minimum and maximum degrees among all vertices of G, respectively. For a subset S of vertices of G, the subgraph of G induced by S is denoted by $G[S]$. A subset S of vertices is an *independent set* if $G[S]$ has no edge. The *independence number*, $\alpha(G)$ of G, is the maximum cardinality among all independent sets.

In [6,7], Fink and Jacobson generalized the concept of independent sets. For a positive integer k, a subset S of vertices of a graph G is *k-independent* if the maximum degree of the subgraph induced by the vertices of S is less than or equal to $k-1$. The *k-independence number* $\alpha_k(G)$ is the maximum cardinality among all k-independent sets in G.

Hedetniemi et al. [12] introduced the concept of alliances in graphs. This concept has been further considered by several other authors, see for example [10,13,16]. Favaron et al. [5] initiated the study of offensive alliances in graphs. Some types of alliance numbers namely global offensive alliance number and global strong offensive alliance number have been considered by Rodríguez-Velazquez et al. [17]. A subset S of vertices of a graph $G = (V, E)$ is a *global offensive alliance* if for every $v \in V - S$, $|N[v] \cap S| \geq |N[v] - S|$. The minimum cardinality among all global offensive alliances of G is called the *global offensive alliance number* of G, and is denoted by $\gamma_o(G)$. A subset S of vertices of a graph $G = (V, E)$ is a *global strong offensive alliance* if for every $v \in V - S$, $|N[v] \cap S| > |N[v] - S|$. The minimum cardinality among all global strong offensive alliances of G is called the *global strong offensive alliance number* of G, and is denoted by $\gamma_{\hat{o}}(G)$. The concept of global offensive alliance in graphs was further studied in, for example, [2,8,9,14,15,18,19]. Balakrishnan et al. [2] showed that the decision problem for global offensive alliance is NP-complete for general graphs. Several upper bounds for the offensive alliance and global offensive alliance numbers are given by Rodríguez-Velazquez et al. [14,15,17] and Harutyunyan [8,9].

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Harutyunyan [9] presented the following probabilistic upper bounds for the global offensive alliance and global strong offensive alliance numbers.

Theorem 1 (*Harutyunyan [9]*). *For any graph $G=(V,E)$ of order n, and $1/2>\alpha>0$,*

$$\gamma_o(G)\le\left(\frac{1}{2}+\alpha\right)n+\left(\frac{1}{2}-\alpha\right)\sum_{v\in V}\exp\left(-\frac{\alpha^2}{1+2\alpha}\deg(v)\right)n.$$

Theorem 2 (*Harutyunyan [9]*). *For any graph $G=(V,E)$ of order n, and $1/2>\alpha>0$,*

$$\gamma_{\hat{o}}(G)\le\left(\frac{1}{2}+\alpha\right)n+\sum_{v\in V}\exp\left(-\frac{\alpha^2}{1+2\alpha}(\deg(v)+1)\right)n.$$

In this paper, we obtain new probabilistic upper bounds for the global offensive alliance number as well as the global strong offensive alliance number of a graph, and improve both Theorems 1 and 2. Our proofs are in similar lines with the proofs of Theorems 1 and 2. The following lower bound on the k-independence number of a graph plays a fundamental role in this paper.

Theorem 3 (*Favaron [4]*). *For every graph G and every positive integer k,*

$$\alpha_k(G)\ge\sum_{v\in V}\frac{k}{1+k\deg(v)}.$$

We also use the well-known Chernoff bound as follows:

Theorem 4 (*Chernoff, [1,3]*). *For any $a>0$ and random variable X that has binomial distribution with probability p and mean pn, $P[X-pn<-a]<e^{\frac{-a^2}{2pn}}$.*

2. New bounds

We first present a new upper bound for the global offensive alliance number of a graph.

Theorem 5. *Let $G=(V,E)$ be a graph of order n, maximum degree Δ and minimum degree δ. For $1/2>\alpha>0$,*

$$\gamma_o(G)\le\left(\frac{1}{2}+\alpha\right)n+\left(\frac{1}{2}-\alpha\right)\sum_{v\in V}\exp\left(-\frac{\alpha^2}{1+2\alpha}\deg(v)\right)n-\frac{n(\lfloor\frac{\delta}{2}\rfloor+1)}{1+(\lfloor\frac{\delta}{2}\rfloor+1)\Delta}\left(\frac{1}{2}+\alpha\right)^{1+\Delta}.$$

Proof. We follow the proof of Theorem 1 given in [9]. Create a subset $S\subseteq V$ by choosing each vertex $v\in V$, independently, with probability $p=1/2+\alpha$. The random set S is going to be part of the global offensive alliance. For every vertex $v\in V$, let X_v denote the number of vertices in the neighborhood of v that are in S. Let

$$Y=\left\{v\in V-S: X_v\le\left\lfloor\frac{\deg(v)}{2}\right\rfloor\right\}.$$

Let $S'=\{v:N[v]\subseteq S\}$, and D be a maximum $(\lfloor\frac{\delta}{2}\rfloor+1)$-independent set in $G[S']$. For any vertex $v\in D$, $\deg_{G[D]}(v)\le\lfloor\frac{\delta}{2}\rfloor$, and so $|N(v)\cap(S-D)|=\deg(v)-\deg_{G[D]}(v)\ge\deg(v)-\lfloor\frac{\delta}{2}\rfloor\ge\frac{\deg(v)}{2}$. Thus $(S-D)\cup Y$ is a global offensive alliance in G. We now estimate the expectation of $|(S-D)\cup Y|$. Clearly, $\mathbb{E}(|(S-D)\cup Y|)\le\mathbb{E}(|S|)+\mathbb{E}(|Y|)-\mathbb{E}(|D|)$, and $\mathbb{E}(|S|)=np$. Note that X_v is a binomial $(\deg(v),p)$ random variable for any vertex $v\in V$. Letting $a=\varepsilon pn$, where $\varepsilon=1-\frac{1}{2p}$, Theorem 4 implies that

$$Pr\left(X_v\le\frac{\deg(v)}{2}\right)=Pr\left(X_v\le(1-\varepsilon)p\deg(v)\right)<e^{\frac{-\varepsilon^2\deg(v)p}{2}}=e^{\frac{-(1-\frac{1}{2p})^2\deg(v)p}{2}}.$$

Then

$$Pr(v\in Y)=Pr(v\notin S)Pr\left(X_v\le\frac{\deg(v)}{2}\right)\le(1-p)e^{-(1-\frac{1}{2p})^2\frac{\deg(v)p}{2}}.$$

Thus, $\mathbb{E}(|Y|)\le\sum_{v\in V}(1-p)e^{\frac{-(1-\frac{1}{2p})^2\deg(v)p}{2}}$. Now

$$\begin{aligned}\mathbb{E}(|S|+|Y|)&\le np+\sum_{v\in V}(1-p)e^{\frac{-(1-\frac{1}{2p})^2\deg(v)p}{2}}\\&=\left(\frac{1}{2}+\alpha\right)n+\left(\frac{1}{2}-\alpha\right)\sum_{v\in V}\exp\left(-\frac{\alpha^2}{1+2\alpha}\deg(v)\right)n.\end{aligned}$$

We next estimate the expectation of $|D|$. By Theorem 3, $|D| \geq \sum_{v\in S'} \frac{(\lfloor\frac{\delta}{2}\rfloor+1)}{1+(\lfloor\frac{\delta}{2}\rfloor+1)\deg_{G[S']}(v)}$. Thus,

$$\begin{aligned}
\mathbb{E}(|D|) &\geq \mathbb{E}\left(\sum_{v\in S'} \frac{(\lfloor\frac{\delta}{2}\rfloor+1)}{1+(\lfloor\frac{\delta}{2}\rfloor+1)\deg_{G[S']}(v)}\right) \\
&\geq \sum_{v\in V} \frac{(\lfloor\frac{\delta}{2}\rfloor+1)}{1+(\lfloor\frac{\delta}{2}\rfloor+1)\deg_{G}(v)} Pr(v\in S') \\
&= \sum_{v\in V} \frac{(\lfloor\frac{\delta}{2}\rfloor+1)}{1+(\lfloor\frac{\delta}{2}\rfloor+1)\deg_{G}(v)} p^{1+\deg_G(v)} \\
&\geq \sum_{v\in V} \frac{(\lfloor\frac{\delta}{2}\rfloor+1)}{1+(\lfloor\frac{\delta}{2}\rfloor+1)\deg_{G}(v)} p^{1+\Delta} \\
&\geq \frac{n(\lfloor\frac{\delta}{2}\rfloor+1)}{1+(\lfloor\frac{\delta}{2}\rfloor+1)\Delta}\left(\frac{1}{2}+\alpha\right)^{1+\Delta}.
\end{aligned}$$

Thus the result follows. ■

We next present a new upper bound for the global strong offensive alliance number of a graph.

Theorem 6. *Let $G=(V,E)$ be a graph of order n, maximum degree Δ and minimum degree δ. For $1/2>\alpha>0$,*

$$\gamma_{\hat{o}}(G) \leq \left(\frac{1}{2}+\alpha\right)n + \sum_{v\in V}\exp\left(-\frac{\alpha^2}{1+2\alpha}(\deg(v)+1)\right)n - \frac{n(\lfloor\frac{\delta-1}{2}\rfloor+1)}{1+(\lfloor\frac{\delta-1}{2}\rfloor+1)\Delta}\left(\frac{1}{2}+\alpha\right)^{1+\Delta}.$$

Proof. We follow the same proof of Theorem 5 given in [9]. Create a subset $S\subseteq V$ by choosing each vertex $v\in V$, independently, with probability $p=1/2+\alpha$. For every vertex $v\in V$, let X_v denote the number of vertices in the neighborhood of v that are in S. Let

$$Y=\left\{v\in V-S : X_v<\left\lfloor\frac{\deg(v)+1}{2}\right\rfloor\right\}.$$

Let $S'=\{v : N[v]\subseteq S\}$, and D be a maximum $(\lfloor\frac{\delta-1}{2}\rfloor+1)$-independent set in $G[S']$. For any vertex $v\in D$, $\deg_{G[D]}(v)\leq\lfloor\frac{\delta-1}{2}\rfloor$, and so $|N(v)\cap(S-D)|=\deg(v)-\deg_{G[D]}(v)\geq\deg(v)-\lfloor\frac{\delta-1}{2}\rfloor>\frac{\deg(v)}{2}$. Thus $(S-D)\cup Y$ is a global strong offensive alliance in G. Clearly, $\mathbb{E}(|S|)=np$. Note that X_v is a binomial $(\deg(v),p)$ random variable for any vertex $v\in V$. Letting $a=\varepsilon pn$, where $\varepsilon=1-\frac{1}{2p}$, Theorem 4 implies that

$$Pr\left(X_v<\frac{\deg(v)+1}{2}\right)=Pr(X_v<(1-\varepsilon)p(\deg(v)+1))<e^{\frac{-\varepsilon^2 p(\deg(v)+1)}{2}}=e^{-(1-\frac{1}{2p})^2 p\frac{(\deg(v)+1)}{2}}.$$

Thus,

$$\begin{aligned}
\mathbb{E}(|S|+|Y|) &\leq np+\sum_{v\in V} e^{\frac{-(1-\frac{1}{2p})^2 p(\deg(v)+1)}{2}} \\
&= \left(\frac{1}{2}+\alpha\right)n+\sum_{v\in V}\exp\left(-\frac{\alpha^2}{1+2\alpha}(\deg(v)+1)\right)n.
\end{aligned}$$

As before, we can see that

$$\mathbb{E}(|D|)\geq\frac{n(\lfloor\frac{\delta-1}{2}\rfloor+1)}{1+(\lfloor\frac{\delta-1}{2}\rfloor+1)\Delta}\left(\frac{1}{2}+\alpha\right)^{1+\Delta}.$$

Thus the result follows. ■

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