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An extended compromise ratio model with an application to reservoir flood control operation under an interval-valued intuitionistic fuzzy environment

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ABSTRACT

This paper presents a novel multiple attribute group decision-making (MAGDM) model based on the compromise ratio method under an interval-valued intuitionistic fuzzy (IVIF) environment. The compromise ratio method under uncertainty is introduced by a group of experts based on the concept that the chosen alternative should be as close as possible to the IVIF-positive-ideal point and as far away from the IVIF-negative-ideal point as possible concurrently. First, an IVIF-weighted geometric averaging (IVIFWGA) operator is employed to aggregate all individual IVIF-decision matrices provided by a group of experts into a collective IVIF-decision matrix. Two new basic IVIF-operations are introduced to handle the evaluation process. Then, an extended collective index in an IVIF environment is proposed to discriminate among alternatives for the evaluation process in terms of subjective and objective information. Finally, to demonstrate the suitability and applicability of the proposed IVIF-MAGDM model, an application example of reservoir flood control operation is given from the recent literature.

1. Introduction

Since the primary work of Zadeh [1], the traditional 0–1 logic was developed to fuzzy logic, described by a membership function between 0 and 1. This development leads to important theoretical extensions and widely used approaches that are successfully applied in many engineering and management fields (e.g., [2–7]). In the last two decades, intuitionistic fuzzy sets (IFSs) were first presented by Atanassov [8] as an extension of Zadeh's fuzzy sets. The IFSs are implemented in numerous industrial applications with remarkable results (e.g., [9–11]). By considering the real-valued membership and non-membership functions represented in interval values, Atanassov and Gargov [12] introduced the notion of the IFSs to interval-valued intuitionistic fuzzy sets (IVIFSs). The basic feature of the IVIFS is that the values of its membership function and non-membership function are intervals rather than exact numbers [13]. Thus, the IVIFSs can properly take into consideration the ambiguity in the information as well as the fuzziness in experts or decision makers (DMs)' preferences and judgments in order to involve different aspects of problems in real-life decision making.

Many real-life decision problems can be taken within the frame of multiple attribute group decision making (MAGDM). The purpose of the MAGDM is to choose an appropriate alternative among a set of alternatives by assessing multiple conflicting attributes by a group of experts. Such decision problems can be solved by using several existing reputable

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MAGDM methods, namely GVIKOR (group Vlse Kriterijuska Optimizacija I Komoromisno Resenje) and GTOPSIS (group technique for order preference by similarity to ideal solution), developed by Opricovic [14] and Hwang and Yoon [15], respectively. Both GVIKOR and GTOPSIS methods are taken into account an aggregation function by characterizing the closeness to the ideal point. These methods can easily structure the problem corresponding to the DMs' needs along with adopted results [16–19].

For the relate literature, some researchers have taken the VIKOR and TOPSIS methods under incomplete and uncertain environment. For instance, Opricovic and Tzeng [20] developed the VIKOR method for analyzing the planning strategies by reducing the future social and economic costs in the area with potential natural hazard. Tzeng et al. [21] applied and compared VIKOR and TOPSIS methods to determine the best compromise alternative fuel model in solving a public transportation problem. Chang and Hsu [22] applied the VIKOR method to determine the best feasible solution according to the selected criteria, including geographical and meteorological factors. The objective of their study was to establish the priority ranking of land-use restrictions in the Tseng-Wen reservoir watershed. Hashemi et al. [23] applied the VIKOR method based on intuitionistic fuzzy sets under multiple criteria to select the potential alternatives in a large-scale water resources development scheme. Vahdani et al. [24] presented the interval-valued fuzzy VIKOR method and built a practical maintenance strategy selection problem to verify their proposed method. Mousavi et al. [3] proposed a stochastic VIKOR method for selection problems in which a group of the DMs described a value for an alternative vs. an attribute by the use of linguistic variables. Opricovic [25] utilized the fuzzy VIKOR method to study the development of a reservoir system for the storage of surface flows of the Mlava River and its tributaries for regional water supply. Liou et al. [26] applied a modified VIKOR method to improve service quality among domestic airlines in Taiwan. Their model allows DMs to understand the gaps between alternatives and aspired-levels in practice. Vinodh et al. [27] selected the VIKOR method for concept selection in the agile manufacturing. Mishra et al. [28] adopted VIKOR method in a fuzzy environment to assess multiple attributes on suppliers' performance and to select the best supplier among a group of alternative suppliers. Girubha and Vinodh [29] used the VIKOR as a decision making tool for the selection of alternate material for instrument panel used in electric car and in order to evaluate this selection process in fuzzy environment. Park et al. [30] extended TOPSIS method to handling MAGDM problems under an IVIF-environment where the information about attributes' weights is partially known. Park et al. [31] developed a procedure for solving MAGDM problems and extended the VIKOR method, in which all the preference information provided by the DMs is presented as IVIF-decision. Yücenur and Demirel [32] extended fuzzy VIKOR method to deal with the criteria and select the most suitable alternative insurance. The VIKOR method focused on ranking from a set of alternatives in the presence of conflicting criteria under fuzzy environment. Vahdani et al. [33] presented a new compromise solution method by a group of experts or the DMs with traditional fuzzy sets to effectively solve the evaluation and selection problems. The method is applied to the contractor selection problem with multi-criteria and multi-judges under uncertainty.

The review of the literature shows that the compromise ratio method under modern fuzzy environment can be introduced as a new research area for solving complex decision problems through the group decision making process. Although the existing approaches have had contributions to decision making under uncertainty, most of the related literature (e.g., [3,5,32,33]) described the individual decision information with traditional fuzzy sets.

This paper presents a novel MAGDM model based on an extended compromise ratio method in an IVIF environment. The major contributions of this paper are given below:

- Two new operations for interval-valued intuitionistic fuzzy sets are presented by taken the operational laws of interval-valued intuitionistic fuzzy numbers (IVIFNs) into consideration.
- In the proposed compromise ratio method, IVIF-ideal separation and anti-ideal separation matrixes are constructed based on the new subtraction operations between IVIFNs to discriminate among the alternatives in the group decision-making problems, unlike the previous studies which were based on traditional fuzzy sets and distances of potential alternatives from the ideal solutions.
- An extended collective index in an IVIF-environment is proposed to rank alternatives according the concept of the relative distance from the IVIF-positive-ideal and IVIF-negative-ideal points and the score and accuracy functions, unlike the previous studies (e.g., [30,31]) which were based on Euclidean distances of each alternative with respect to the reference points and Ref. [33] with traditional fuzzy numbers.
- The model has the ability to reflect both subjective judgment and objective information in real-life applications under an IVIF environment. Characteristics of alternatives and decision attributes are represented by linguistic terms and then are converted into IVIFNs.

It is worth to mention that the compromise ratio method presented by the authors [33] is employed in this paper for the evaluation process; however, the main contributions and differences of the proposed model combined with a modern fuzzy set in an IVIF-form are explained above unlike the previous studies (e.g., [30,31]). Furthermore, application example from the recent literature is examined for the reservoir flood control operation to demonstrate the implementation process of the IVIF-MAGDM model. The model can assist the DMs to make their efficient decisions for solving intricate decision problems under uncertainty.

A limitation of the proposed IVIF-MAGDM as a generalized decision making model under uncertainty is that it employs a new subtraction operation for calculating the distance between two IVIFNs. This operation is valid under some conditions. In fact, these conditions should be met through the decision process. In some states, the experts or DMs may be asked to somewhat modify their opinions. Moreover, another limitation is values of the weights for the strategy of the majority attributes (λ and γ) in the proposed model; these values are regarded as important input parameters for the ranking step. Failing to choose proper values might change the output of the proposed IVIF-MAGDM model remarkably (i.e., the ranking of alternatives). Hence, the values of λ and γ provided by the experts should be properly considered in the proposed model.

The rest of this paper is organized as follows. The basic concepts, definitions and notations of VIKOR method and IVIFNs are introduced in Section 2. In Section 3, a new IVIF-MAGDM model is proposed based on the combination of the concepts of compromise ratio method and IVIFNs. In Section 4, an illustrative example, applying the proposed IVIF-MAGDM model to evaluate alternatives of the reservoir flood control operation is presented, after which this paper discusses how the new IVIF-MAGDM model is effective. Finally, conclusions and future research are presented in Section 6.

2. Preliminaries

2.1. VIKOR method for group decision making

Opricovic and Tzeng [20,34] introduced VIKOR method for complex systems which means multi-criteria optimization and compromise solution. This method focuses on ranking and selecting from a set of alternatives, and determines compromise solutions for a problem with non-commensurable and conflicting criteria, which can help the DMs to reach a final decision. Here, the compromise solution is a feasible solution which is the closest to the ideal solution point, and a compromise means an agreement established by mutual concessions [35].

Suppose the MAGDM problem that has m decision alternatives, $A_1, A_2, \ldots, A_m$, n attributes, $C_1, C_2, \ldots, C_n$, and l experts or DMs, $E_1, E_2, \ldots, E_l$. The structure of the decision matrix for kth expert can be expressed as follows:

$$F^k = [f_{ij}^k] = \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_m \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \cdots & C_n \end{array} \\ \begin{bmatrix} f_{11}^k & f_{12}^k & \cdots & f_{1n}^k \\ f_{21}^k & f_{22}^k & \cdots & f_{2n}^k \\ \vdots & \vdots & \ddots & \vdots \\ f_{m1}^k & f_{m2}^k & \cdots & f_{mn}^k \end{bmatrix} \end{array},$$

f_{ij}^k, $k = 1,2,\ldots,l$, $i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$, is the performance rating value of the ith alternative vs. the jth attribute that is evaluated by kth expert, and let w_j^k be the relative weight vector about the attributes that is provided by kth expert. Then, the performance rating and weight values that are provided by experts, can be aggregated by the arithmetic mean method, $f_{ij} = \frac{1}{l}\sum_{k=1}^{l} f_{ij}^k$ and $w_j = \frac{1}{l}\sum_{k=1}^{l} w_j^k$.

After the aggregation process, for alternative A_i, the rating of the jth attribute and the relative weight of each attribute are denoted by f_{ij} and W_j, respectively. Also, the best and worst values are regarded as f_j^* and f_j^-, respectively. The main VIKOR method for decision making can be summarized as follows [20,34]:

Step 1: The best f_j^* and the worst f_j^- values of all criteria functions are calculated $j = 1, 2, \ldots, n$. For the jth function as the benefit, we have:

$$f_j^* = \max_i f_{ij} \tag{1}$$

and

$$f_j^- = \min_i f_{ij}. \tag{2}$$

Step 2: The values S_i and R_i are provided; $i = 1, 2, \ldots, m$, by these relations:

$$S_i = \sum_{j=1}^{n} w_j(f_j^* - f_{ij})/(f_j^* - f_j^-), \tag{3}$$

$$R_i = \max_j w_j(f_j^* - f_{ij})/(f_j^* - f_j^-), \tag{4}$$

where are the weights of criteria representing their relative importance. Also, S_i represent the individual regrets/gaps, and R_i represent the maximum individual gaps.

Step 3: The values Q_i are computed by the relation as follows, $i = 1, 2, \ldots, m$:

$$Q_i = \lambda(S_i - S^*)/(S^- - S^*) + (1 - \lambda)(R_i - R^*)/(R^- - R^*), \tag{5}$$

where

$$S^* = \min_i S_i, \quad S^- = \max_i S_i, \tag{6}$$

$$R^* = \min_i R_i, \quad R^- = \max_i R_i, \tag{7}$$

λ demonstrates the weight of the strategy for the majority of attributes which is assumed to be $\lambda = 0.5$. $\mathrm{Min}_i S_i$ emphasizes the minimization of the average sum of the individual regrets/gaps, and $\min_i R_i$ describes the minimization of the maximum individual regret/gaps for prioritizing the improvement. In fact, S^* is the minimum value of S_i, which is the maximum group utility, and R^* is the minimum value of R_i, which is the minimum individual regret of the opponent. Also, Q is the ranking index and it is provided according to the group utility and individual regret of the opponent.

Step 4: The alternatives are ranked by values of S, R and Q.
Step 5: Compromise solution the alternative A' is obtained and then it is ranked as the best by the measure Q (minimum) if two conditions are satisfied as follows:

C_1. Acceptable advantage:

$$Q(A'') - Q(A') \geqslant DQ, \tag{8}$$

where A'' is an alternative with second position in the ranking list by Q; $DQ = 1/(m-1)$; m is the number of alternatives.
C_2. Acceptable stability in decision-making:
Alternative A' must also be the best ranked by S or/and R. This compromise solution is stable within a decision-making process which could be voting by majority rule (when $\lambda > 0.5$ is needed), or by consensus $\lambda \approx 0.5$, or with veto ($\lambda < 0.5$).
If one of the conditions is not satisfied, then a set of compromise solutions is proposed which consists of:

- Alternatives A' and A'' if only condition C_2 is not satisfied, or
- Alternatives $A', A'', \ldots, A^{(M)}$ if condition C_1 is not satisfied; $A^{(M)}$ is determined by the relation $Q(A^{(M)}) - Q(A') < DQ$ for maximum M (the positions of these alternatives are in closeness).

The best alternative, ranked by Q, is the one with the minimum value of Q. The main ranking result is the compromise ranking list of alternatives and the compromise solution with the advantage rate.

2.2. Basic concepts and operations of interval-valued intuitionistic fuzzy sets

Let a set X be fixed, an IVIFS in X is defined as [12]:

$$\tilde{A} = \left\{ \langle x, \bar{\mu}_{\tilde{A}}(x), \bar{v}_{\tilde{A}}(x) \rangle | x \in X \right\}, \tag{9}$$

where $\bar{\mu}_{\tilde{A}}(x) \subset [0,1]$, $\bar{v}_{\tilde{A}}(x) \subset [0,1]$, $x \in X$ and $\sup \bar{\mu}_{\tilde{A}}(x) + \sup \bar{v}_{\tilde{A}}(x) \leqslant 1$, $\forall x \in X$. Especially, if $\inf \bar{\mu}_{\tilde{A}}(x) = \sup \bar{\mu}_{\tilde{A}}(x)$ and $\inf \bar{v}_{\tilde{A}}(x) = \sup \bar{v}_{\tilde{A}}(x)$, then the IVIFS $\tilde{A}$ is reduced to an IFS.

For convenience, an IVIFS $\tilde{A}$ is denoted by $\left\langle \left[\mu^L_{\tilde{A}}(x), \mu^U_{\tilde{A}}(x)\right], \left[v^L_{\tilde{A}}(x), v^U_{\tilde{A}}(x)\right] \right\rangle$, where $\left[\mu^L_{\tilde{A}}(x), \mu^U_{\tilde{A}}(x)\right] \subset [0,1]$, $\left[v^L_{\tilde{A}}(x), v^U_{\tilde{A}}(x)\right] \subset [0,1]$, $\mu^U_{\tilde{A}}(x) + v^U_{\tilde{A}}(x) \leqslant 1$ and for each element x we can calculate the hesitancy degree of an interval-valued intuitionistic fuzzy of $x \in X$ in $\tilde{A}$ defined as follows:

$$\bar{\pi}_{\tilde{A}} = \left[1 - \mu^U_{\tilde{A}}(x) - v^U_{\tilde{A}}(x), 1 - \mu^L_{\tilde{A}}(x) - v^L_{\tilde{A}}(x)\right]. \tag{10}$$

Atanassov and Gargov [12] and Atanassov [36] first introduced some basic operations on IVIFSs, which not only can ensure that the operational results are IVIFSs but also are suitable for the calculus of variables under the IVIF-environment. Motivated by the operations in Atanassov and Gargov [12], Atanassov [36], and Xu [37] defined four operational laws of IVIFNs, which can be employed in this paper, as follows:

Let $\tilde{a} = \langle [\mu^L_{\tilde{a}}, \mu^U_{\tilde{a}}], [v^L_{\tilde{a}}, v^U_{\tilde{a}}] \rangle$ and $\tilde{b} = \left\langle \left[\mu^L_{\tilde{b}}, \mu^U_{\tilde{b}}\right], \left[v^L_{\tilde{b}}, v^U_{\tilde{b}}\right] \right\rangle$ be any two IVIFNs, then

$$\tilde{a} \oplus \tilde{b} = \left\langle \left[\mu^L_{\tilde{a}} + \mu^L_{\tilde{b}} - \mu^L_{\tilde{a}} \cdot \mu^L_{\tilde{b}}, \mu^U_{\tilde{a}} + \mu^U_{\tilde{b}} - \mu^U_{\tilde{a}} \cdot \mu^U_{\tilde{b}}\right], \left[v^L_{\tilde{a}} \cdot v^L_{\tilde{b}}, v^U_{\tilde{a}} \cdot v^U_{\tilde{b}}\right] \right\rangle, \tag{11}$$

$$\tilde{a} \otimes \tilde{b} = \left\langle \left[\mu^L_{\tilde{a}} \cdot \mu^L_{\tilde{b}}, \mu^U_{\tilde{a}} \cdot \mu^U_{\tilde{b}}\right], \left[v^L_{\tilde{a}} + v^L_{\tilde{b}} - v^L_{\tilde{a}} \cdot v^L_{\tilde{b}}, v^U_{\tilde{a}} + v^U_{\tilde{b}} - v^U_{\tilde{a}} \cdot v^U_{\tilde{b}}\right] \right\rangle, \tag{12}$$

$$\lambda\tilde{a} = \left\langle \left[1 - (1 - \mu^L_{\tilde{a}})^{\lambda}, 1 - (1 - \mu^U_{\tilde{a}})^{\lambda}\right], \left[(v^L_{\tilde{a}})^{\lambda}, (v^U_{\tilde{a}})^{\lambda}\right] \right\rangle, \quad \lambda > 0, \tag{13}$$

$$\tilde{a}^{\lambda} = \left\langle \left[(\mu^L_{\tilde{a}})^{\lambda}, (\mu^U_{\tilde{a}})^{\lambda}\right], \left[1 - (1 - v^L_{\tilde{a}})^{\lambda}, 1 - (1 - v^U_{\tilde{a}})^{\lambda}\right] \right\rangle, \quad \lambda > 0, \tag{14}$$

which can ensure the operational results are also IVIFNs.

Yu et al. [38] defined a score function S to measure an IVIFN $\tilde{a}$ as follows:

$$S(\tilde{a}) = \frac{1}{4}\left[2 + \mu^L_{\tilde{a}} - v^L_{\tilde{a}} + \mu^U_{\tilde{a}} - v^U_{\tilde{a}}\right], \tag{15}$$

where $S(\tilde{a}) \in [0,1]$. The larger the value of $S(\tilde{a})$, the higher the IVIFN $\tilde{a}$ is. Especially, if $S(\tilde{a}) = 1$, then $\tilde{a} = \langle [1,1], [0,0] \rangle$, which is the largest IVIFN; if $S(\tilde{a}) = 0$, then $\tilde{a} = \langle [0,0], [1,1] \rangle$, which is the smallest IVIFN.

In addition, Ye [39] defined an accuracy function H to evaluate the accuracy degree of an IVIFN as follows:

$$H(\tilde{a}) = \mu_{\tilde{a}}^L + \mu_{\tilde{a}}^U - 1 + \frac{\nu_{\tilde{a}}^L + \nu_{\tilde{a}}^U}{2}, \tag{16}$$

where $H(\tilde{a}) \in [-1, 1]$. The larger the value of $H(\tilde{a})$, the higher the accuracy degree of the IVIFN $\tilde{a}$ is.The relation between the score function S and the accuracy function H of an IVIFN is similar to the relation between mean and variance in statistics. On the basis of the score function S and the accuracy function H, in the following, an order relation between IVIFNs is provided [36].

Definition 1. Let $\tilde{a} = \langle [\mu_{\tilde{a}}^L, \mu_{\tilde{a}}^U], [\nu_{\tilde{a}}^L, \nu_{\tilde{a}}^U] \rangle$ and $\tilde{b} = \left\langle \left[\mu_{\tilde{b}}^L, \mu_{\tilde{b}}^U\right], \left[\nu_{\tilde{b}}^L, \nu_{\tilde{b}}^U\right] \right\rangle$ be any two IVIFNs, then:

- If $S(\tilde{a}) < S(\tilde{b})$, then $\tilde{a}$ is smaller than $\tilde{b}$, denoted by $\tilde{a} < \tilde{b}$;
- if $S(\tilde{a}) = S(\tilde{b})$, then
  1. If $H(\tilde{a}) = H(\tilde{b})$, then $\tilde{a}$ and $\tilde{b}$ represent the same information, which denotes indifference between $\tilde{a}$ and $\tilde{b}$, defined as $\tilde{a} = \tilde{b}$;
  2. if $H(\tilde{a}) < H(\tilde{b})$, then $\tilde{a}$ is smaller than $\tilde{b}$, defined as $\tilde{a} < \tilde{b}$.

Definition 2. Let $\tilde{a} = \langle [\mu_{\tilde{a}}^L, \mu_{\tilde{a}}^U], [\nu_{\tilde{a}}^L, \nu_{\tilde{a}}^U] \rangle$ and $\tilde{b} = \left\langle \left[\mu_{\tilde{b}}^L, \mu_{\tilde{b}}^U\right], \left[\nu_{\tilde{b}}^L, \nu_{\tilde{b}}^U\right] \right\rangle$ be any two IVIFNs, then we have [36]:

$$\mu_{\tilde{a}}^L \leqslant \mu_{\tilde{b}}^L,\ \mu_{\tilde{a}}^U \leqslant \mu_{\tilde{b}}^U,\ \nu_{\tilde{a}}^L \geqslant \nu_{\tilde{b}}^L \text{ and } \nu_{\tilde{a}}^U \geqslant \nu_{\tilde{b}}^U \ \Rightarrow\ \tilde{a} \leqslant \tilde{b}.$$

Definition 3. Let $\tilde{\alpha}_j$ ($j = 1,2,\ldots,n$) be a collection of IVIFNs, The geometric aggregation operator of the IVIFNs is computed by [40]:

$$IVIFWGA_\omega(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = \prod_{j=1}^n \tilde{\alpha}_j^{\omega_j} = \left\langle \left[\prod_{j=1}^n \left(\mu_{\tilde{\alpha}_j}^L\right)^{\omega_j}, \prod_{j=1}^n \left(\mu_{\tilde{\alpha}_j}^U\right)^{\omega_j}\right], \left[1 - \prod_{j=1}^n \left(1 - \nu_{\tilde{\alpha}_j}^L\right)^{\omega_j}, 1 - \prod_{j=1}^n \left(1 - \nu_{\tilde{\alpha}_j}^U\right)^{\omega_j}\right] \right\rangle, \tag{17}$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight vector of $\tilde{\alpha}_j$ ($j = 1,2,\ldots,n$), $\omega_j \in [0,1]$, and $\sum_{j=1}^n \omega_j = 1$.

Theorem 1. *Let* $\tilde{a} = \langle [\mu_{\tilde{a}}^L, \mu_{\tilde{a}}^U], [\nu_{\tilde{a}}^L, \nu_{\tilde{a}}^U] \rangle$, $\tilde{b} = \left\langle \left[\mu_{\tilde{b}}^L, \mu_{\tilde{b}}^U\right], \left[\nu_{\tilde{b}}^L, \nu_{\tilde{b}}^U\right] \right\rangle$ *and* $\tilde{c} = \langle [\mu_{\tilde{c}}^L, \mu_{\tilde{c}}^U], [\nu_{\tilde{c}}^L, \nu_{\tilde{c}}^U] \rangle$ *be three IVIFNs; the following equation*

$$\tilde{c} = \tilde{a} - \tilde{b} = \left\langle \left[\frac{\mu_{\tilde{a}}^L - \mu_{\tilde{b}}^L}{1 - \mu_{\tilde{b}}^L}, \frac{\mu_{\tilde{a}}^U - \mu_{\tilde{b}}^U}{1 - \mu_{\tilde{b}}^U}\right], \left[\frac{\nu_{\tilde{a}}^L}{\nu_{\tilde{b}}^L}, \frac{\nu_{\tilde{a}}^U}{\nu_{\tilde{b}}^U}\right] \right\rangle, \tag{18}$$

is valid under the following conditions

$$\tilde{a} \geqslant \tilde{b}, \quad \left[\mu_{\tilde{b}}^L, \mu_{\tilde{b}}^U\right] \neq [1, 1], \quad \left[\nu_{\tilde{b}}^L, \nu_{\tilde{b}}^U\right] \neq [0, 0] \quad \text{and} \quad \mu_{\tilde{a}}^U \cdot \nu_{\tilde{b}}^U - \mu_{\tilde{b}}^U \cdot \nu_{\tilde{a}}^U \leqslant \nu_{\tilde{b}}^U - \nu_{\tilde{a}}^U.$$

Proof. Let us consider an equation of the type:

$$\tilde{c} \oplus \tilde{b} = \tilde{a}, \tag{19}$$

where the IVIFNs $\tilde{a}$ and $\tilde{b}$ are provided, and the problem is to obtain the unknown IVIFN $\tilde{c}$ which satisfies $\left[\mu_{\tilde{c}}^L, \mu_{\tilde{c}}^U\right] \subset [0, 1]$, $\left[\nu_{\tilde{c}}^L, \nu_{\tilde{c}}^U\right] \subset [0, 1]$ and $\mu_{\tilde{c}}^U + \nu_{\tilde{c}}^U \leqslant 1$. By using Eq. (11), we know that

$$\begin{aligned}
&\mu_{\tilde{c}}^L + \mu_{\tilde{b}}^L - \mu_{\tilde{c}}^L \cdot \mu_{\tilde{b}}^L = \mu_{\tilde{a}}^L,\\
&\mu_{\tilde{c}}^U + \mu_{\tilde{b}}^U - \mu_{\tilde{c}}^U \cdot \mu_{\tilde{b}}^U = \mu_{\tilde{a}}^U,\\
&\nu_{\tilde{c}}^L \cdot \nu_{\tilde{b}}^L = \nu_{\tilde{a}}^L,\\
&\nu_{\tilde{c}}^U \cdot \nu_{\tilde{b}}^U = \nu_{\tilde{a}}^U.
\end{aligned} \tag{20}$$

Then,

$$\begin{aligned}
&\mu_{\tilde{c}}^L = \frac{\mu_{\tilde{a}}^L - \mu_{\tilde{b}}^L}{1 - \mu_{\tilde{b}}^L},\\
&\mu_{\tilde{c}}^U = \frac{\mu_{\tilde{a}}^U - \mu_{\tilde{b}}^U}{1 - \mu_{\tilde{b}}^U},\\
&\nu_{\tilde{c}}^L = \frac{\nu_{\tilde{a}}^L}{\nu_{\tilde{b}}^L},\\
&\nu_{\tilde{c}}^U = \frac{\nu_{\tilde{a}}^U}{\nu_{\tilde{b}}^U}.
\end{aligned} \tag{21}$$

Unfortunately, $\tilde{c}$ with relations of Eq. (21) may not be an IVIFN. The membership degree of $\tilde{c}$ should take values in the interval [0,1], i.e., $0 \leqslant \frac{\mu_{\tilde{a}}^L - \mu_{\tilde{b}}^L}{1-\mu_{\tilde{b}}^L} \leqslant 1$ and $0 \leqslant \frac{\mu_{\tilde{a}}^U - \mu_{\tilde{b}}^U}{1-\mu_{\tilde{b}}^U} \leqslant 1$.

The right-hand sides of both conditions are valid because $\mu_{\tilde{a}}^L \leqslant 1$ and $\mu_{\tilde{a}}^U \leqslant 1$, but the left-hand sides are correct in the cases that $\mu_{\tilde{a}}^L \geqslant \mu_{\tilde{b}}^L$, $\mu_{\tilde{a}}^U \geqslant \mu_{\tilde{b}}^U$, $\mu_{\tilde{b}}^L \neq 1$ and $\mu_{\tilde{b}}^U \neq 1$.

Similarly, non-membership degree of $\tilde{c}$ should take values in the interval [0,1], i.e., $0 \leqslant \frac{v_{\tilde{a}}^L}{v_{\tilde{b}}^L} \leqslant 1$ and $0 \leqslant \frac{v_{\tilde{a}}^U}{v_{\tilde{b}}^U} \leqslant 1$.

It is obvious that the left-hand sides of these conditions are correct because $v_{\tilde{a}}^L \geqslant 0$, $v_{\tilde{b}}^L \geqslant 0$, $v_{\tilde{a}}^U \geqslant 0$ and $v_{\tilde{b}}^U \geqslant 0$, but the right-hand sides of them are correct in the cases that $v_{\tilde{a}}^L \leqslant v_{\tilde{b}}^L$, $v_{\tilde{a}}^U \leqslant v_{\tilde{b}}^U$, $v_{\tilde{b}}^L \neq 0$ and $v_{\tilde{b}}^U \neq 0$.

Integrating $\mu_{\tilde{a}}^L \geqslant \mu_{\tilde{b}}^L$, $\mu_{\tilde{a}}^U \geqslant \mu_{\tilde{b}}^U$, $v_{\tilde{a}}^L \leqslant v_{\tilde{b}}^L$ and $v_{\tilde{a}}^U \leqslant v_{\tilde{b}}^U$, we know that $\tilde{a} \geqslant \tilde{b}$ based on Definition 2. Hence, the above-mentioned inequalities hold only if $\tilde{a} \geqslant \tilde{b}$, $\left[\mu_{\tilde{b}}^L, \mu_{\tilde{b}}^U\right] \neq [1,1]$ and $\left[v_{\tilde{b}}^L, v_{\tilde{b}}^U\right] \neq [0,0]$.

Furthermore, $\tilde{c}$ is an IVIFN and thus, $\mu_{\tilde{c}}^U + v_{\tilde{c}}^U = \frac{\mu_{\tilde{a}}^U - \mu_{\tilde{b}}^U}{1-\mu_{\tilde{b}}^U} + \frac{v_{\tilde{a}}^U}{v_{\tilde{b}}^U} = \frac{\mu_{\tilde{a}}^U \cdot v_{\tilde{b}}^U - \mu_{\tilde{b}}^U \cdot v_{\tilde{b}}^U + v_{\tilde{a}}^U - \mu_{\tilde{b}}^U \cdot v_{\tilde{a}}^U}{v_{\tilde{b}}^U - \mu_{\tilde{b}}^U \cdot v_{\tilde{b}}^U} \leqslant 1$.

Based on the above-mentioned conditions, we know that $\mu_{\tilde{b}}^U < 1$ *and* $v_{\tilde{b}}^U \neq 0$; hence, $v_{\tilde{b}}^U - \mu_{\tilde{b}}^U \cdot v_{\tilde{b}}^U > 0$.

Then, $\mu_{\tilde{a}}^U \cdot v_{\tilde{b}}^U - \mu_{\tilde{b}}^U \cdot v_{\tilde{a}}^U \leqslant v_{\tilde{b}}^U - v_{\tilde{a}}^U$

which completes the proof of Theorem 1. □

Theorem 2. *Let* $\tilde{a} = \langle [\mu_{\tilde{a}}^L, \mu_{\tilde{a}}^U], [v_{\tilde{a}}^L, v_{\tilde{a}}^U] \rangle$, $\tilde{b} = \left\langle \left[\mu_{\tilde{b}}^L, \mu_{\tilde{b}}^U\right], \left[v_{\tilde{b}}^L, v_{\tilde{b}}^U\right] \right\rangle$ *and* $\tilde{d} = \left\langle \left[\mu_{\tilde{d}}^L, \mu_{\tilde{d}}^U\right], \left[v_{\tilde{d}}^L, v_{\tilde{d}}^U\right] \right\rangle$ *be three IVIFNs; the following equation*

$$\tilde{d} = \tilde{a} \div \tilde{b} = \left\langle \left[\frac{\mu_{\tilde{a}}^L}{\mu_{\tilde{b}}^L}, \frac{\mu_{\tilde{a}}^U}{\mu_{\tilde{b}}^U}\right], \left[\frac{v_{\tilde{a}}^L - v_{\tilde{b}}^L}{1 - v_{\tilde{b}}^L}, \frac{v_{\tilde{a}}^U - v_{\tilde{b}}^U}{1 - v_{\tilde{b}}^U}\right] \right\rangle \tag{22}$$

is valid under the following conditions $\tilde{a} \leqslant \tilde{b}$, $\left[\mu_{\tilde{b}}^L, \mu_{\tilde{b}}^U\right] \neq [0,0]$, $\left[v_{\tilde{b}}^L, v_{\tilde{b}}^U\right] \neq [1,1]$ and $\mu_{\tilde{a}}^U \cdot v_{\tilde{b}}^U - \mu_{\tilde{b}}^U \cdot v_{\tilde{a}}^U \geqslant \mu_{\tilde{a}}^U - \mu_{\tilde{b}}^U$.

Proof. Let us consider an equation of the type:

$$\tilde{d} \otimes \tilde{b} = \tilde{a}, \tag{23}$$

where the IVIFNs $\tilde{a}$ and $\tilde{b}$ are given, and the problem is to find the unknown IVIFN $\tilde{d}$ which satisfies $\left[\mu_{\tilde{d}}^L, \mu_{\tilde{d}}^U\right] \subset [0,1]$, $\left[v_{\tilde{d}}^L, v_{\tilde{d}}^U\right] \subset [0,1]$ and $\mu_{\tilde{d}}^U + v_{\tilde{d}}^U \leqslant 1$. Using Eq. (12), we know that

$$\begin{aligned} &\mu_{\tilde{d}}^L \cdot \mu_{\tilde{b}}^L = \mu_{\tilde{a}}^L, \\ &\mu_{\tilde{d}}^U \cdot \mu_{\tilde{b}}^U = \mu_{\tilde{a}}^U, \\ &v_{\tilde{d}}^L + v_{\tilde{b}}^L - v_{\tilde{d}}^L \cdot v_{\tilde{b}}^L = v_{\tilde{a}}^L, \\ &v_{\tilde{d}}^U + v_{\tilde{b}}^U - v_{\tilde{d}}^U \cdot v_{\tilde{b}}^U = v_{\tilde{a}}^U. \end{aligned} \tag{24}$$

Then,

$$\begin{aligned} &\mu_{\tilde{d}}^L = \frac{\mu_{\tilde{a}}^L}{\mu_{\tilde{b}}^L}, \\ &\mu_{\tilde{d}}^U = \frac{\mu_{\tilde{a}}^U}{\mu_{\tilde{b}}^U}, \\ &v_{\tilde{d}}^L = \frac{v_{\tilde{a}}^L - v_{\tilde{b}}^L}{1 - v_{\tilde{b}}^L}, \\ &v_{\tilde{d}}^U = \frac{v_{\tilde{a}}^U - v_{\tilde{b}}^U}{1 - v_{\tilde{b}}^U}. \end{aligned} \tag{25}$$

Unfortunately, $\tilde{d}$ with relations of Eq. (25) may not be an IVIFN. The membership degree of $\tilde{d}$ should take values in the interval [0,1], i.e., $0 \leqslant \frac{\mu_{\tilde{a}}^L}{\mu_{\tilde{b}}^L} \leqslant 1$ and $0 \leqslant \frac{\mu_{\tilde{a}}^U}{\mu_{\tilde{b}}^U} \leqslant 1$.

It is obvious that the left-hand sides of both conditions are valid because $\mu_{\tilde{a}}^L \geqslant 0$, $\mu_{\tilde{b}}^L \geqslant 0$, $\mu_{\tilde{a}}^U \geqslant 0$ and $\mu_{\tilde{b}}^U \geqslant 0$, but the right-hand sides are correct in the cases that $\mu_{\tilde{a}}^L \leqslant \mu_{\tilde{b}}^L$, $\mu_{\tilde{a}}^U \leqslant \mu_{\tilde{b}}^U$, $\mu_{\tilde{b}}^L \neq 0$ and $\mu_{\tilde{b}}^U \neq 0$.

Similarly, non-membership degree of $\tilde{d}$ should take values in the interval [0,1], i.e., $0 \leqslant \frac{v_{\tilde{a}}^L - v_{\tilde{b}}^L}{1-v_{\tilde{b}}^L} \leqslant 1$ and $0 \leqslant \frac{v_{\tilde{a}}^U - v_{\tilde{b}}^U}{1-v_{\tilde{b}}^U} \leqslant 1$.

The right-hand sides of these conditions are valid because $v_{\tilde{a}}^L \leqslant 1$ and $v_{\tilde{a}}^U \leqslant 1$, but the left-hand sides are correct in the cases that $v_{\tilde{a}}^L \geqslant v_{\tilde{b}}^L$, $v_{\tilde{a}}^U \geqslant v_{\tilde{b}}^U$, $v_{\tilde{b}}^L \neq 1$ and $v_{\tilde{b}}^U \neq 1$.

Integrating $\mu_{\tilde{a}}^L \leqslant \mu_{\tilde{b}}^L$, $\mu_{\tilde{a}}^U \leqslant \mu_{\tilde{b}}^U$, $v_{\tilde{a}}^L \geqslant v_{\tilde{b}}^L$ and $v_{\tilde{a}}^U \geqslant v_{\tilde{b}}^U$, we know that $\tilde{a} \leqslant \tilde{b}$ based on Definition 2. Hence, the above-mentioned inequalities hold only if $\tilde{a} \leqslant \tilde{b}$, $\left[\mu_{\tilde{b}}^L, \mu_{\tilde{b}}^U\right] \neq [0,0]$ and $\left[v_{\tilde{b}}^L, v_{\tilde{b}}^U\right] \neq [1,1]$.

Furthermore, $\tilde{d}$ is an IVIFN and thus, $\mu_{\tilde{d}}^U + \nu_{\tilde{d}}^U = \frac{\mu_{\tilde{a}}^U}{\mu_{\tilde{b}}^U} + \frac{\nu_{\tilde{a}}^U - \nu_{\tilde{b}}^U}{1-\nu_{\tilde{b}}^U} = \frac{\mu_{\tilde{a}}^U - \mu_{\tilde{a}}^U \cdot \nu_{\tilde{b}}^U + \mu_{\tilde{b}}^U \cdot \nu_{\tilde{a}}^U - \mu_{\tilde{b}}^U \cdot \nu_{\tilde{b}}^U}{\mu_{\tilde{b}}^U - \mu_{\tilde{b}}^U \cdot \nu_{\tilde{b}}^U} \leqslant 1$.

Based on the above-mentioned conditions, we know that $\nu_{\tilde{b}}^U < 1$ and $\mu_{\tilde{b}}^U \neq 0$; hence, $\mu_{\tilde{b}}^U - \mu_{\tilde{b}}^U \cdot \nu_{\tilde{b}}^U > 0$. Then, $\mu_{\tilde{a}}^U \cdot \nu_{\tilde{b}}^U - \mu_{\tilde{b}}^U \cdot \nu_{\tilde{a}}^U \geqslant \mu_{\tilde{a}}^U - \mu_{\tilde{b}}^U$ which completes the proof of Theorem 2. □

As above-mentioned, $\tilde{c}$ and $\tilde{d}$ may not be IVIFNs in Theorems 1 and 2, because they may not satisfy the IVIFN conditions, presented on relation (9). In these states, values of $\tilde{a}$ and $\tilde{b}$ should be modified so that $\tilde{c}$ and $\tilde{d}$ will be IVIFNs, if possible. Otherwise, we can employ the distance operator presented by Park [30] instead of the subtraction operator for $\tilde{c}$. Also, for $\tilde{d}$ we can convert $\tilde{a}$ and $\tilde{b}$ to crisp numbers by the score or accuracy functions (i.e., relations (15) and (16)) when the conditions of the division operator do not satisfy.

3. Proposed novel interval-valued intuitionistic fuzzy MAGDM model

Sets and input parameters:

A: set of alternatives, $A = \{A_1, A_2, \ldots, A_m\}$
C: set of conflicting attributes, $C = \{C_1, C_2, \ldots, C_n\}$
E: set of experts or DMs, $E = \{E_1, E_2, \ldots, E_l\}$
$\bar{\mu}$: interval value of membership degree of IVIFNs
$\bar{\nu}$: interval value of non-membership degree of IVIFNs
$\tilde{x}_{ij}^{(k)}$: IVIF-performance rating of the ith alternative A_i with respect to the jth attribute C_j provided by kth expert
$a_{ij}^{(k)}$: lower bound of $\bar{\mu}$ of performance rating $\tilde{x}_{ij}^{(k)}$ provided by kth expert
$b_{ij}^{(k)}$: upper bound of $\bar{\mu}$ of performance rating $\tilde{x}_{ij}^{(k)}$ provided by kth expert
$c_{ij}^{(k)}$: lower bound of $\bar{\nu}$ of performance rating $\tilde{x}_{ij}^{(k)}$ provided by kth expert
$d_{ij}^{(k)}$: upper bound of $\bar{\nu}$ of performance rating $\tilde{x}_{ij}^{(k)}$ provided by kth expert
$\tilde{x}_{ij}$: IVIF-aggregated performance rating
a_{ij}: lower bound of $\bar{\mu}$ of aggregated performance rating $\tilde{x}_{ij}$
b_{ij}: Upper bound of $\bar{\mu}$ of aggregated performance rating $\tilde{x}_{ij}$
c_{ij}: lower bound of $\bar{\nu}$ of aggregated performance rating $\tilde{x}_{ij}$
d_{ij}: upper bound of $\bar{\nu}$ of aggregated performance rating $\tilde{x}_{ij}$
$\tilde{\xi}^{(k)}$: IVIF-relative importance of kth expert
$\xi_1^{(k)}$: lower bound of $\bar{\mu}$ of relative importance $\tilde{\xi}^{(k)}$ provided by kth expert
$\xi_2^{(k)}$: upper bound of $\bar{\mu}$ of relative importance $\tilde{\xi}^{(k)}$ provided by kth expert
$\xi_3^{(k)}$: lower bound of $\bar{\nu}$ of relative importance $\tilde{\xi}^{(k)}$ provided by kth expert
$\xi_4^{(k)}$: upper bound of $\bar{\nu}$ of relative importance $\tilde{\xi}^{(k)}$ provided by kth expert
$\tilde{W}_j^{(k)}$: IVIF-weight of attribute j provided by kth expert
$w_{j1}^{(k)}$: lower bound of $\bar{\mu}$ of weight $\tilde{W}_j^{(k)}$
$w_{j2}^{(k)}$: upper bound of $\bar{\mu}$ of weight $\tilde{W}_j^{(k)}$
$w_{j3}^{(k)}$: lower bound of $\bar{\nu}$ of weight $\tilde{W}_j^{(k)}$
$w_{j4}^{(k)}$: upper bound of $\bar{\nu}$ of weight $\tilde{W}_j^{(k)}$
$\tilde{W}_j$: IVIF-aggregated weight of attribute j
w_{j1}: lower bound of $\bar{\mu}$ of aggregated weight $\tilde{W}_j$
w_{j2}: upper bound of $\bar{\mu}$ of aggregated weight $\tilde{W}_j$
w_{j3}: lower bound of $\bar{\nu}$ of aggregated weight $\tilde{W}_j$
w_{j4}: upper bound of $\bar{\nu}$ of aggregated weight $\tilde{W}_j$
$\tilde{r}_{ij}$: performance rating of ith alternative A_i vs. the jth attribute Cj
$\tilde{r}_j^*$: IVIF-positive-ideal solution of the jth attribute Cj
a_j^*: lower bound of $\bar{\mu}$ of positive-ideal solution $\tilde{r}_j^*$
b_j^*: upper bound of $\bar{\mu}$ of positive-ideal solution $\tilde{r}_j^*$
c_j^*: lower bound of $\bar{\nu}$ of positive-ideal solution $\tilde{r}_j^*$
d_j^*: upper bound of $\bar{\nu}$ of positive-ideal solution $\tilde{r}_j^*$
$\tilde{r}_j^-$: IVIF-negative-ideal solution of the jth attribute Cj
a_j^-: lower bound of $\bar{\mu}$ of negative-ideal solution $\tilde{r}_j^-$
b_j^-: upper bound of $\bar{\mu}$ of negative-ideal solution $\tilde{r}_j^-$
c_j^-: lower bound of $\bar{\nu}$ of negative-ideal solution $\tilde{r}_j^-$
d_j^-: upper bound of $\bar{\nu}$ of negative-ideal solution $\tilde{r}_j^-$
$\tilde{f}_{ij}^*$: IVIF-distance between $\tilde{r}_{ij}$ and $\tilde{r}_j^*$
$\tilde{f}_{ij}^-$: IVIF-distance between $\tilde{r}_{ij}$ and $\tilde{r}_j^-$
λ, γ: weight for the strategy of the majority attributes

The proposed IVIF-MAGDM model under multiple attributes deals with uncertain and imprecise data and information. This decision problem takes into account performance values of all candidates vs. multiple conflicting attributes as well as the weight of attributes. The evaluation values are expressed as linguistic terms and they are convertible to IVIFNs during the decision process under uncertainty by Tables 1 and 2.

For the MAGDM problem, let $E=\{E_1, E_2, \ldots, E_l\}$ be the set of the experts or DMs, $A=\{A_1,A_2,\ldots,A_m\}$ be a finite set of alternatives, and $C=\{C_1, C_2, \ldots, C_n\}$ be the set of conflicting attributes.

The characteristic of the candidate A_i is represented by an IVIFN as follows:

$$\tilde{A}_i=\left\{\left\langle C_j,\left[\mu^L_{\tilde{A}_i}(C_j),\mu^U_{\tilde{A}_i}(C_j)\right],\left[\nu^L_{\tilde{A}_i}(C_j),\nu^U_{\tilde{A}_i}(C_j)\right]\right\rangle C_j\in C\right\}, \tag{26}$$

where $0\leqslant\mu^U_{\tilde{A}_i}(C_j)+\nu^U_{\tilde{A}_i}(C_j)\leqslant 1,\ \mu^L_{\tilde{A}_i}(C_j)\geqslant 0,\ \nu^L_{\tilde{A}_i}(C_j)\geqslant 0, j=1,2,\ldots,n,\ i=1,2,\ldots,m.$

The IVIFN that is the pair of intervals $\bar{\mu}^{(k)}_{\tilde{A}_i}(C_j)=\left[a^{(k)}_{ij},b^{(k)}_{ij}\right]$, $\bar{\nu}^{(k)}_{\tilde{A}_i}(C_j)=\left[c^{(k)}_{ij},d^{(k)}_{ij}\right]$ for $C_j\in C$ is denoted by $\tilde{x}^{(k)}_{ij}=\left\langle\left[a^{(k)}_{ij},b^{(k)}_{ij}\right],\left[c^{(k)}_{ij},d^{(k)}_{ij}\right]\right\rangle$, where $\left[a^{(k)}_{ij},b^{(k)}_{ij}\right]$ indicates the degree that the candidate A_i satisfies the attribute C_j provided by the expert or DM E_k $(k=1,2,\ldots,l)$, $\left[c^{(k)}_{ij},d^{(k)}_{ij}\right]$ indicates the degree that the candidate A_i does not satisfies the attribute C_j given by the expert E_k.

The rating of each alternative vs. the objective attributes must be transformed to an IVIFN. Hence, first they are normalized into the range of [0,1]. The normalized rating for them can be calculated as follows.

For benefit attributes, we have:

$$\hat{x}^{(k)}_{ij}=\frac{x^{(k)}_{ij}-\min\limits_i\left\{x^{(k)}_{ij}\right\}}{\max\limits_i\left\{x^{(k)}_{ij}\right\}-\min\limits_i\left\{x^{(k)}_{ij}\right\}},$$

and for cost attributes, we have:

$$\hat{x}^{(k)}_{ij}=\frac{\max\limits_i\left\{x^{(k)}_{ij}\right\}-x^{(k)}_{ij}}{\max\limits_i\left\{x^{(k)}_{ij}\right\}-\min\limits_i\left\{x^{(k)}_{ij}\right\}}.$$

Then, $\hat{x}^{(k)}_{ij}$ can be written as an IVIFN $\tilde{x}^{(k)}_{ij}=\left\langle\bar{\mu}^{(k)}_{\tilde{x}_{ij}},\bar{\nu}^{(k)}_{\tilde{x}_{ij}}\right\rangle=\left\langle\left[\hat{x}^{(k)}_{ij},\hat{x}^{(k)}_{ij}\right],\left[1-\hat{x}^{(k)}_{ij},1-\hat{x}^{(k)}_{ij}\right]\right\rangle$. Thus, both subjective and objective attributes in the proposed model can be simultaneously considered and handled to solve complex decision making problems.

$X^{(k)}=\left(\tilde{x}^{(k)}_{ij}\right)_{m\times n}$ provided by the expert E_k as an IVIF-decision matrix is obtained as the following form:

Table 1
Linguistic terms for the rating of alternatives.

Linguistic terms	Interval-valued intuitionistic fuzzy numbers
Very good (VG)/very high (VH)	⟨[0.80, 0.90], [0.05, 0.10]⟩
Good (G)/high (H)	⟨[0.55, 0.70], [0.10, 0.20]⟩
Medium good (MG)/medium high (MH)	⟨[0.45, 0.60], [0.15, 0.30]⟩
Fair (F)/medium (M)	⟨[0.30, 0.50], [0.20, 0.40]⟩
Medium bad (MB)/medium low (ML)	⟨[0.25, 0.40], [0.35, 0.50]⟩
Bad (B)/low (L)	⟨[0.10, 0.30], [0.45, 0.60]⟩
Very bad (VB)/very low (VL)	⟨[0.00, 0.10], [0.70, 0.90]⟩

Table 2
Linguistic terms for the relative importance of the DMs and criteria.

Linguistic terms	Interval-valued intuitionistic fuzzy numbers
Very important (VI)	⟨[0.80, 0.90], [0.05, 0.10]⟩
Important (I)	⟨[0.60, 0.75], [0.10, 0.20]⟩
Medium (M)	⟨[0.30, 0.50], [0.25, 0.45]⟩
Unimportant (UI)	⟨[0.20, 0.35], [0.45, 0.60]⟩
Very unimportant (VUI)	⟨[0.00, 0.10], [0.70, 0.90]⟩

$$X^{(k)}=\left(\tilde{x}_{ij}^{(k)}\right)_{m\times n}=\begin{array}{c} \\ A_1\\ A_2\\ \vdots\\ A_m\end{array}\left|\begin{array}{cccc} C_1 & C_2 & \ldots & C_n\\ \left\langle[a_{11}^{(k)},b_{11}^{(k)}],[c_{11}^{(k)},d_{11}^{(k)}]\right\rangle & \left\langle[a_{12}^{(k)},b_{12}^{(k)}],[c_{12}^{(k)},d_{12}^{(k)}]\right\rangle & \ldots & \left\langle[a_{1n}^{(k)},b_{1n}^{(k)}],[c_{1n}^{(k)},d_{1n}^{(k)}]\right\rangle\\ \left\langle[a_{21}^{(k)},b_{21}^{(k)}],[c_{21}^{(k)},d_{21}^{(k)}]\right\rangle & \left\langle[a_{22}^{(k)},b_{22}^{(k)}],[c_{22}^{(k)},d_{22}^{(k)}]\right\rangle & \ldots & \left\langle[a_{2n}^{(k)},b_{2n}^{(k)}],[c_{2n}^{(k)},d_{2n}^{(k)}]\right\rangle\\ \vdots & \vdots & \vdots & \vdots\\ \left\langle[a_{m1}^{(k)},b_{m1}^{(k)}],[c_{m1}^{(k)},d_{m1}^{(k)}]\right\rangle & \left\langle[a_{m2}^{(k)},b_{m2}^{(k)}],[c_{m2}^{(k)},d_{m2}^{(k)}]\right\rangle & \ldots & \left\langle[a_{mn}^{(k)},b_{mn}^{(k)}],[c_{mn}^{(k)},d_{mn}^{(k)}]\right\rangle\end{array}\right|.$$

According to above-mentioned descriptions, steps of the proposed novel IVIF-MAGDM model are presented as follows

Step 1: A committee of the experts or DMs (E_k, $k = 1,2,\ldots,l$) is established to determine the best alternative among a set of potential alternatives (candidates) by considering the conflicting attributes.

Step 2: The relative importance of each DM is determined as a linguistic term and is convertible to an IVIFN $\left(\tilde{\xi}^{(k)}=\left\langle\bar{\mu}_{\tilde{\xi}}^{(k)},\bar{v}_{\tilde{\xi}}^{(k)}\right\rangle=\left\langle\left[\xi_1^{(k)},\xi_2^{(k)}\right],\left[\xi_3^{(k)},\xi_4^{(k)}\right]\right\rangle\right)$.

Step 3: Proper subjective and objective attributes are identified for the selection problem.

Step 4: The weight of each selected attribute j by kth DM is subjectively described by a linguistic term and is transformed into the IVIFN $\left(\tilde{W}_j^{(k)}=\left\langle\bar{\mu}_{\tilde{W}_j}^{(k)},\bar{v}_{\tilde{W}_j}^{(k)}\right\rangle=\left\langle\left[w_{j1}^{(k)},w_{j2}^{(k)}\right],\left[w_{j3}^{(k)},w_{j4}^{(k)}\right]\right\rangle\right)$.

Step 5: The aggregated IVIF-weight of each selected attribute based on the interval-valued intuitionistic fuzzy weighted geometric averaging (IVIFWGA) operator is calculated, $IVIFWGA_\omega(\tilde{W}_1,\tilde{W}_2,\ldots,\tilde{W}_l)$, by:

$$\tilde{W}_j=\left\langle\bar{\mu}_{\tilde{W}_j},\bar{v}_{\tilde{W}_j}\right\rangle=\left\langle[w_{j1},w_{j2}],[w_{j3},w_{j4}]\right\rangle$$
$$=\left\langle\left[\prod_{k=1}^{l}(\xi_1^{(k)}.w_{j1}^{(k)})^{\omega_k},\prod_{k=1}^{l}(\xi_2^{(k)}.w_{j2}^{(k)})^{\omega_k}\right],\left[1-\prod_{k=1}^{l}\left(1-\xi_3^{(k)}-w_{j3}^{(k)}+\xi_3^{(k)}.w_{j3}^{(k)}\right)^{\omega_k},1-\prod_{k=1}^{l}\left(1-\xi_4^{(k)}-w_{j4}^{(k)}+\xi_4^{(k)}.w_{j4}^{(k)}\right)^{\omega_k}\right]\right\rangle \tag{27}$$

where $\omega=(\omega_1,\omega_2,\ldots,\omega_l)^T=(\frac{1}{l},\frac{1}{l},\ldots,\frac{1}{l})^T$ is the weight vector of $\tilde{W}_j$ ($j = 1,2,\ldots,n$), $\omega_k \in [0,1]$, and $\sum_{k=1}^{l}\omega_k=1$.

Step 6: The performance rating of each potential alternative vs. the selected attributes is evaluated by each DM ($\tilde{x}_{ij}^{(k)}$).

Step 7: The IVIF-performance matrix is formed for each DM ($X^{(k)}$).

Step 8: The aggregated IVIF-decision matrix is constructed based on opinions of the DMs and the IVIFWGA operator $\left(IVIFWGA_\omega\left(\tilde{x}_{ij}^{(1)},\tilde{x}_{ij}^{(2)},\ldots,\tilde{x}_{ij}^{(l)}\right)\right)$ by:

$$\tilde{x}_{ij}=\left\langle\bar{\mu}_{\tilde{x}_{ij}},\bar{v}_{\tilde{x}_{ij}}\right\rangle=\langle[a_{ij},b_{ij}],[c_{ij},d_{ij}]\rangle$$
$$=\left\langle\left[\prod_{k=1}^{l}\left(\xi_1^{(k)}.a_{ij}^{(k)}\right)^{\omega_k},\prod_{k=1}^{l}\left(\xi_2^{(k)}.b_{ij}^{(k)}\right)^{\omega_k}\right],\left[1-\prod_{k=1}^{l}\left(1-\xi_3^{(k)}-c_{ij}^{(k)}+\xi_3^{(k)}.c_{ij}^{(k)}\right)^{\omega_k},1-\prod_{k=1}^{l}\left(1-\xi_4^{(k)}-d_{ij}^{(k)}+\xi_4^{(k)}.d_{ij}^{(k)}\right)^{\omega_k}\right]\right\rangle, \tag{28}$$

where $\omega=(\omega_1,\omega_2,\ldots,\omega_l)^T=(\frac{1}{l},\frac{1}{l},\ldots,\frac{1}{l})^T$ is the weight vector of $\tilde{x}_{ij}^{(k)}$ ($i = 1,2,\ldots,m$; $j = 1,2,\ldots,n$; $k = 1,2,\ldots,l$), $\omega_k \in [0,1]$, and $\sum_{k=1}^{l}\omega_k=1$.

Step 9: The weighted aggregated IVIF-decision matrix is determined by considering the different importance of attributes as follows:

$$R=[\tilde{r}_{ij}]_{m\times n}, \tag{29}$$

where,

$$\tilde{r}_{ij}=\tilde{W}_j\otimes\tilde{x}_{ij}=\left\langle[w_{j1}.a_{ij},w_{j2}.b_{ij}],[w_{j3}+c_{ij}-w_{j3}.c_{ij},w_{j4}+d_{ij}-w_{j4}.d_{ij}]\right\rangle, \tag{30}$$

Step 10: IVIF-positive-ideal and IVIF-negative-ideal solutions, defined as A^* and A^-, are determined by:

$$A^*=\{\tilde{r}_1^*,\ldots,\tilde{r}_n^*\}^T=\left\{\left(\max_i\tilde{r}_{ij}\,|j\in\Omega_B\right),\left(\min_i\tilde{r}_{ij}\,|j\in\Omega_C\right)|i=1,2,\ldots,m\right\}^T, \tag{31}$$

$$A^-=\{\tilde{r}_1^-,\ldots,\tilde{r}_n^-\}^T=\left\{\left(\min_i\tilde{r}_{ij}\,|j\in\Omega_B\right),\left(\max_i\tilde{r}_{ij}\,|j\in\Omega_C\right)|i=1,2,\ldots,m\right\}^T, \tag{32}$$

where Ω_B is associated with benefit attributes (i.e., the larger A_i, the greater preference), and Ω_C is associated with cost attributes (i.e., the smaller A_i, the greater preference). $\tilde{r}_j^*=\left\langle\left[a_j^*,b_j^*\right],\left[c_j^*,d_j^*\right]\right\rangle$ and $\tilde{r}_j^-=\left\langle\left[a_j^-,b_j^-\right],\left[c_j^-,d_j^-\right]\right\rangle$, $j = 1,2,\ldots,n$.

Step 11: IVIF-positive-ideal separation matrix (F^*) is defined as follows:

$$F^* = \left[\tilde{f}^*_{ij}\right] = \begin{bmatrix} \tilde{r}^*_1 - \tilde{r}_{11} & \tilde{r}^*_2 - \tilde{r}_{12} & \cdots & \tilde{r}^*_n - \tilde{r}_{1n} \\ \tilde{r}^*_1 - \tilde{r}_{21} & \tilde{r}^*_2 - \tilde{r}_{22} & \cdots & \tilde{r}^*_n - \tilde{r}_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{r}^*_1 - \tilde{r}_{m1} & \tilde{r}^*_2 - \tilde{r}_{m2} & \cdots & \tilde{r}^*_n - \tilde{r}_{mn} \end{bmatrix} \tag{33}$$

for benefit attributes, and

$$F^* = \left[\tilde{f}^*_{ij}\right] = \begin{bmatrix} \tilde{r}_{11} - \tilde{r}^*_1 & \tilde{r}_{12} - \tilde{r}^*_2 & \cdots & \tilde{r}_{1n} - \tilde{r}^*_n \\ \tilde{r}_{21} - \tilde{r}^*_1 & \tilde{r}_{22} - \tilde{r}^*_2 & \cdots & \tilde{r}_{2n} - \tilde{r}^*_n \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{r}_{m1} - \tilde{r}^*_1 & \tilde{r}_{m2} - \tilde{r}^*_2 & \cdots & \tilde{r}_{mn} - \tilde{r}^*_n \end{bmatrix} \tag{34}$$

for cost attributes.

Also, IVIF-negative-ideal separation matrix (F^-) is defined as follows:

$$F^- = \left[\tilde{f}^-_{ij}\right] = \begin{bmatrix} \tilde{r}_{11} - \tilde{r}^-_1 & \tilde{r}_{12} - \tilde{r}^-_2 & \cdots & \tilde{r}_{1n} - \tilde{r}^-_n \\ \tilde{r}_{21} - \tilde{r}^-_1 & \tilde{r}_{22} - \tilde{r}^-_2 & \cdots & \tilde{r}_{2n} - \tilde{r}^-_n \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{r}_{m1} - \tilde{r}^-_1 & \tilde{r}_{m2} - \tilde{r}^-_2 & \cdots & \tilde{r}_{mn} - \tilde{r}^-_n \end{bmatrix} \tag{35}$$

for benefit attributes, and

$$F^- = \left[\tilde{f}^-_{ij}\right] = \begin{bmatrix} \tilde{r}^-_1 - \tilde{r}_{11} & \tilde{r}^-_2 - \tilde{r}_{12} & \cdots & \tilde{r}^-_n - \tilde{r}_{1n} \\ \tilde{r}^-_1 - \tilde{r}_{21} & \tilde{r}^-_2 - \tilde{r}_{22} & \cdots & \tilde{r}^-_n - \tilde{r}_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{r}^-_1 - \tilde{r}_{m1} & \tilde{r}^-_2 - \tilde{r}_{m2} & \cdots & \tilde{r}^-_n - \tilde{r}_{mn} \end{bmatrix} \tag{36}$$

for cost attributes.

Step 12: $\tilde{\Re}_i$, $\tilde{\Im}_i$, $\tilde{\zeta}_i$ and $\tilde{\xi}_i$ values are computed for $i = 1, 2, \ldots, m$ by Eqs. (37)–(40) as follows:

$$\tilde{\Re}_i = \sum_{j=1}^{n} \tilde{f}^*_{ij}, \tag{37}$$

$$\tilde{\Im}_i = \max_j \tilde{f}^*_{ij}, \tag{38}$$

$$\tilde{\zeta}_i = \sum_{j=1}^{n} \tilde{f}^-_{ij}, \tag{39}$$

$$\tilde{\xi}_i = \max_j \tilde{f}^-_{ij}. \tag{40}$$

$\Re_i$, $\Im_i$, ζ_i and ξ_i values are calculated based on the accuracy function by Eqs. (41)–(48) as follows:

$$\Re_i = 1 - \left[\prod_{j=1}^{n}\left(1 - \frac{a^*_j - w_{j1}\cdot a_{ij}}{1 - w_{j1}\cdot a_{ij}}\right) + \prod_{j=1}^{n}\left(1 - \frac{b^*_j - w_{j2}\cdot b_{ij}}{1 - w_{j2}\cdot b_{ij}}\right)\right] + \frac{1}{2}\left[\prod_{j=1}^{n}\left(\frac{c^*_j}{w_{j3} + c_{ij} - w_{j3}\cdot c_{ij}}\right) + \prod_{j=1}^{n}\left(\frac{d^*_j}{w_{j4} + d_{ij} - w_{j4}\cdot d_{ij}}\right)\right], \tag{41}$$

$$\Im_i = \max_j\left(\frac{a^*_j - w_{j1}\cdot a_{ij}}{1 - w_{j1}\cdot a_{ij}} + \frac{b^*_j - w_{j2}\cdot b_{ij}}{1 - w_{j2}\cdot b_{ij}} - 1 + \frac{1}{2}\left(\frac{c^*_j}{w_{j3} + c_{ij} - w_{j3}\cdot c_{ij}} + \frac{d^*_j}{w_{j4} + d_{ij} - w_{j4}\cdot d_{ij}}\right)\right), \tag{42}$$

$$\zeta_i = 1 - \left[\prod_{j=1}^{n}\left(1 - \frac{w_{j1}\cdot a_{ij} - a^-_j}{1 - a^-_j}\right) + \prod_{j=1}^{n}\left(1 - \frac{w_{j2}\cdot b_{ij} - b^-_j}{1 - b^-_j}\right)\right] + \frac{1}{2}\left[\prod_{j=1}^{n}\left(\frac{w_{j3} + c_{ij} - w_{j3}\cdot c_{ij}}{c^-_j}\right) + \prod_{j=1}^{n}\left(\frac{w_{j4} + d_{ij} - w_{j4}\cdot d_{ij}}{d^-_j}\right)\right], \tag{43}$$

$$\xi_i = \max_j\left(\frac{w_{j1}\cdot a_{ij} - a_j^-}{1-a_j^-} + \frac{w_{j2}\cdot b_{ij} - b_j^-}{1-b_j^-} - 1 + \frac{1}{2}\left(\frac{w_{j3} + c_{ij} - w_{j3}\cdot c_{ij}}{c_j^-} + \frac{w_{j4} + d_{ij} - w_{j4}\cdot d_{ij}}{d_j^-}\right)\right) \tag{44}$$

for benefit attributes, and

$$\begin{aligned}\Re_i = 1 - &\left[\prod_{j=1}^{n}\left(1 - \frac{w_{j1}\cdot a_{ij} - a_j^*}{1-a_j^*}\right) + \prod_{j=1}^{n}\left(1 - \frac{w_{j2}\cdot b_{ij} - b_j^*}{1-b_j^*}\right)\right] \\ &+ \frac{1}{2}\left[\prod_{j=1}^{n}\left(\frac{w_{j3} + c_{ij} - w_{j3}\cdot c_{ij}}{c_j^*}\right) + \prod_{j=1}^{n}\left(\frac{w_{j4} + d_{ij} - w_{j4}\cdot d_{ij}}{d_j^*}\right)\right],\end{aligned} \tag{45}$$

$$\Im_i = \max_j\left(\frac{w_{j1}\cdot a_{ij} - a_j^*}{1-a_j^*} + \frac{w_{j2}\cdot b_{ij} - b_j^*}{1-b_j^*} - 1 + \frac{1}{2}\left(\frac{w_{j3} + c_{ij} - w_{j3}\cdot c_{ij}}{c_j^*} + \frac{w_{j4} + d_{ij} - w_{j4}\cdot d_{ij}}{d_j^*}\right)\right), \tag{46}$$

$$\begin{aligned}\zeta_i = 1 - &\left[\prod_{j=1}^{n}\left(1 - \frac{a_j^- - w_{j1}\cdot a_{ij}}{1-w_{j1}\cdot a_{ij}}\right) + \prod_{j=1}^{n}\left(1 - \frac{b_j^- - w_{j2}\cdot b_{ij}}{1-w_{j2}\cdot b_{ij}}\right)\right] \\ &+ \frac{1}{2}\left[\prod_{j=1}^{n}\left(\frac{c_j^-}{w_{j3} + c_{ij} - w_{j3}\cdot c_{ij}}\right) + \prod_{j=1}^{n}\left(\frac{d_j^-}{w_{j4} + d_{ij} - w_{j4}\cdot d_{ij}}\right)\right],\end{aligned} \tag{47}$$

$$\xi_i = \max_j\left(\frac{a_j^- - w_{j1}\cdot a_{ij}}{1-w_{j1}\cdot a_{ij}} + \frac{b_j^- - w_{j2}\cdot b_{ij}}{1-w_{j2}\cdot b_{ij}} - 1 + \frac{1}{2}\left(\frac{c_j^-}{w_{j3} + c_{ij} - w_{j3}\cdot c_{ij}} + \frac{d_j^-}{w_{j4} + d_{ij} - w_{j4}\cdot d_{ij}}\right)\right) \tag{48}$$

for cost attributes.

Step 13: The values of indices τ_i and η_i are calculated. Two indexes are obtained by [33]:

$$\tau_i = \begin{cases}\frac{\Im_i - \Im^+}{\Im^- - \Im^+} & \text{if } \Re^- = \Re^+ \\ \frac{\Re_i - \Re^+}{\Re^- - \Re^+} & \text{if } \Im^- = \Im^+ \\ \left(\frac{\Re_i - \Re^+}{\Re^- - \Re^+}\right)\lambda + \left(\frac{\Im_i - \Im^+}{\Im^- - \Im^+}\right)(1-\lambda) & \text{Otherwise}\end{cases} \tag{49}$$

and

$$\eta_i = \begin{cases}\frac{\xi_i - \xi^-}{\xi^+ - \xi^-} & \text{if } \zeta^+ = \zeta^-, \\ \frac{\zeta_i - \zeta^-}{\zeta^+ - \zeta^-} & \text{if } \xi^+ = \xi^-, \\ \left(\frac{\xi_i - \xi^-}{\xi^+ - \xi^-}\right)\gamma + \left(\frac{\zeta_i - \zeta^-}{\zeta^+ - \zeta^-}\right)(1-\gamma) & \text{otherwise,}\end{cases} \tag{50}$$

where $\begin{cases}\Re^+ = \min_i \Re_i \\ \Re^- = \max_i \Re_i\end{cases}$, $\begin{cases}\Im^+ = \min_i \Im_i \\ \Im^- = \max_i \Im_i\end{cases}$, $\begin{cases}\zeta^+ = \max_i \zeta_i \\ \zeta^- = \min_i \zeta_i\end{cases}$, $\begin{cases}\xi^+ = \max_i \xi_i \\ \xi^- = \min_i \xi_i\end{cases}$, λ and γ are regarded as a weight for the strategy of the majority attributes, whereas $(1-\lambda)$ and $(1-\gamma)$ are the weights of the individual regret. The values of λ and γ fall within the range of 0–1, and these strategies can be compromised by $\lambda = 0.5$ and $\gamma = 0.5$.

Step 14: Collective index (CI) is calculated by:

$$CI_i = \tau_i + \frac{1}{\eta_{i(B)}} + \phi_{i'}, \tag{51}$$

where the second term refers to all i for which $\eta_i > 0$ while $\phi_{i'}$ refers to all i' for which $\eta_i = 0$ and $\phi_{i'} = \left(\min \eta_{i(B)_i}\right)^{\min_j w_j}$, where $\min_j w_j = \min_j\left(w_{j1} + w_{j2} - 1 + \frac{w_{j3} + w_{j4}}{2}\right)$.

Step 15: The alternatives are ranked according to the preference order. The best alternative can be obtained by the preference rank order of τ_i and η_i. The minimum value of the CI demonstrates the better performance for the alternative i.

4. Application of the proposed model for reservoir flood control operation

In this section, the proposed IVIF-MAGDM model is applied to evaluate the operation alternatives of a reservoir by an application example from the recent literature [41]. Generally speaking, the reservoir flood control operation is complex in nature, dealing with a variety of qualitative attributes arising from environmental, social and even political concerns. In addition, by considering imprecision in information and subjectivity in human opinions, the decision making for the reservoir flood control operation problems can be taken within the frame of the MAGDM under fuzzy environments to create a flexible tool for the DMs in this engineering application. Flood control aims to decrease the flood peak discharge during the flood season and at the same time keep the water level of the reservoir as low as possible at the end of this flood. Hence, this paper takes the following several attributes into consideration for the evaluation [41]:

C_1 – flood peak discharge at downstream
C_2 – the less difference between the design flood level and the highest water level during the operation
C_3 – sediment load in reservoir area
C_4 – the risk of flooding in the downstream protected regions
C_5 – the risk of failure of the dam and its structures

There are five feasible operation alternatives A_1, A_2, A_3, A_4 and A_5 for a major flood after preliminary screening. In the following, the presented IVIF-MAGDM model is employed to evaluate and select the best alternative. The values of three quantitative attributes are calculated for each of the operation alternatives. These values are shown in the form of IVIFNs to take account of the uncertainties in the modeling process. For each qualitative attribute, each expert or DM applies the linguistic values to evaluate each alternative as given in Table 1. The weight of five selected attributes and the DMs are described by using the following linguistic terms: very unimportant (VUI), unimportant (UI), medium (M), important (I) and very important (VI), which are defined in Table 2.

A committee of three professional experts or DMs (E_1, E_2 and E_3) is formed to conduct the evaluation and to select the most suitable operation alternatives (Step 1). Due to their different backgrounds and experience, each expert is assigned a weight based on their importance in the engineering application to reservoir flood control operation. These weights are provided by the top manager at the beginning of the group decision making process, which are represented as IVIFNs as given in Table 3 (Step 2).

The weight of each selected attribute by three DMs is subjectively described by linguistic terms and is transformed into the IVIFNs (Step 4). The IVIF-weight of each selected attribute is aggregated based on the IVIFWGA operator ($IVIFWGA_{\omega}(\tilde{W}_1, \tilde{W}_2, \ldots, \tilde{W}_l)$) by Eq. (27) as provided in Table 4 (Step 5).

The fuzzy ratings of five operation alternatives by the linguistic variables, and their respective IVIFNs in Table 1 are evaluated by the DMs with respect to qualitative attributes. Then, the IVIF-performance matrix is formed for each of three DMs as illustrated in Table 5 (Steps 6 and 7). Aggregated IVIF-decision matrix is constructed by Eq. (28). The obtained results are provided in Table 6 based on the IVIFWGA operator ($\mathrm{IVIFWGA}_{\omega}\left(\tilde{x}_{ij}^{(1)}, \tilde{x}_{ij}^{(2)}, \ldots, \tilde{x}_{ij}^{(l)}\right)$) (Step 8).

After calculating the aggregated IVIF-decision matrix and the weights of five selected attributes, the weighted IVIF-decision matrix is constructed and given in Table 7 (Step 9). According to the concept of IVIFNs, the IVIF-positive-ideal and IVIF-negative-ideal solutions are computed according to the benefit attribute (i.e., C_2) and the cost attributes (i.e., C_1, C_3, C_4 and C_5) in this application example (Step 10).

IVIF-positive-ideal separation matrix ($\tilde{F}^{*}$) and IVIF-negative-ideal separation matrix ($\tilde{F}^{-}$) are computed by Eqs. (33)–(36) as follows (Step 11):

$$F_{i1}^{*} = \begin{bmatrix} \langle[0.3074, 0.5012], [0.3816, 0.4694]\rangle \\ \langle[0.1844, 0.3007], [0.5788, 0.6816]\rangle \\ \langle[0.0000, 0.0000], [1.0000, 1.0000]\rangle \\ \langle[0.1229, 0.2005], [0.6415, 0.7878]\rangle \\ \langle[0.3688, 0.6014], [0.2830, 0.3633]\rangle \end{bmatrix}, \quad F_{i2}^{*} = \begin{bmatrix} \langle[0.0341, 0.0872], [0.8508, 0.8967]\rangle \\ \langle[0.0861, 0.1822], [0.6969, 0.7882]\rangle \\ \langle[0.1053, 0.2110], [0.6496, 0.7561]\rangle \\ \langle[0.1327, 0.2594], [0.5609, 0.7031]\rangle \\ \langle[0.0000, 0.0000], [1.0000, 1.0000]\rangle, \end{bmatrix}$$

Table 3
Relative importance of the DMs.

	E_1	E_2	E_3
Linguistic terms	Medium	Important	Very important
Interval-valued intuitionistic fuzzy number	$\langle[0.30, 0.50], [0.25, 0.45]\rangle)$	$\langle[0.60, 0.75], [0.10, 0.20]\rangle$	$\langle[0.80, 0.90], [0.05, 0.10]\rangle$

Table 4
Interval-valued intuitionistic fuzzy weights of the criteria and their aggregations.

	C_1	C_2	C_3	C_4	C_5
E_1	VI	VI	M	M	M
E_2	VI	I	I	VI	I
E_3	I	M	M	I	I
Aggregated interval-valued weight	$\langle[0.7268, 0.8469], [0.0670, 0.1347]\rangle$	$\langle[0.5241, 0.6962], [0.1377, 0.2657]\rangle$	$\langle[0.3780, 0.5724], [0.2030, 0.3768]\rangle$	$\langle\langle[0.5241, 0.6962], [0.1377, 0.2657]\rangle$	$\langle[0.4762, 0.6552], [0.1531, 0.2939]\rangle$

Table 5
Ratings of operation alternatives vs. the selected criteria.

Criteria	Operation alternatives	Experts		
		E_1	E_2	E_3
C_4	A_1	M	M	MH
	A_2	MH	H	M
	A_3	MH	M	MH
	A_4	L	VL	ML
	A_5	VH	H	H
C_5	A_1	M	M	MH
	A_2	MH	H	MH
	A_3	MH	ML	M
	A_4	M	M	ML
	A_5	H	H	MH

Table 6
Aggregated interval-valued intuitionistic fuzzy decision matrix.

Alter.	Criteria				
	C_1	C_2	C_3	C_4	C_5
A_1	⟨[0.8636, 0.9545], [0.0250, 0.0455]⟩	⟨[0.9306, 0.9583], [0.0280, 0.0417]⟩	⟨[0.6316, 0.7368], [0.1579, 0.2632]⟩	⟨[0.1800, 0.3699], [0.2960, 0.5362]⟩	⟨[0.1800, 0.3699], [0.2960, 0.5362]⟩
A_2	⟨[0.7727, 0.8636], [0.0750, 0.1364]⟩	⟨[0.8750, 0.9028], [0.0694, 0.0972]⟩	⟨[0.7895, 0.8947], [0.0630, 0.1053]⟩	⟨[0.2203, 0.4138], [0.2679, 0.4895]⟩	⟨[0.2522, 0.4398], [0.2529, 0.4626]⟩
A_3	⟨[0.6364, 0.7273], [0.1818, 0.2727]⟩	⟨[0.8528, 0.8833], [0.0861, 0.1167]⟩	⟨[0.7368, 0.8421], [0.0840, 0.1579]⟩	⟨[0.2060, 0.3931], [0.2817, 0.5117]⟩	⟨[0.1694, 0.3434], [0.3431, 0.5635]⟩
A_4	⟨[0.7273, 0.8182], [0.0909, 0.1818]⟩	⟨[0.8194, 0.8472], [0.1250, 0.1528]⟩	⟨[0.8947, 1.0000], [0.0000, 0.0000]⟩	⟨[0.0000, 0.1594], [0.5903, 0.8007]⟩	⟨[0.1480, 0.3232], [0.3563, 0.5854]⟩
A_5	⟨[0.9091, 1.0000], [0.0000, 0.0000]⟩	⟨[0.9639, 1.0000], [0.0000, 0.0000]⟩	⟨[0.6842, 0.7895], [0.1053, 0.2105]⟩	⟨[0.3266, 0.5300], [0.2098, 0.3890]⟩	⟨[0.2696, 0.4630], [0.2385, 0.4381]⟩

Table 7
Weighted aggregated interval-valued intuitionistic fuzzy decision matrix.

Alter.	Criteria				
	C_1	C_2	C_3	C_4	C_5
A_1	⟨[0.6277, 0.8084], [0.0903, 0.1740]⟩	⟨[0.4877, 0.6672], [0.1618, 0.2963]⟩	⟨[0.2387, 0.4217], [0.3288, 0.5408]⟩	⟨[0.0943, 0.2576], [0.3930, 0.6594]⟩	⟨[0.0857, 0.2424], [0.4038, 0.6725]⟩
A_2	⟨[0.5617, 0.7314], [0.1369, 0.2527]⟩	⟨[0.4586, 0.6285], [0.1975, 0.3371]⟩	⟨[0.2984, 0.5121], [0.2532, 0.4424]⟩	⟨[0.1155, 0.2881], [0.3687, 0.6251]⟩	⟨[0.1201, 0.2881], [0.3677, 0.6205]⟩
$A3$	⟨[0.4625, 0.6160], [0.2366, 0.3707]⟩	⟨[0.4470, 0.6150], [0.2119, 0.3513]⟩	⟨[0.2785, 0.4820], [0.2700, 0.4752]⟩	⟨[0.1080, 0.2737], [0.3806, 0.6414]⟩	⟨[0.0807, 0.2250], [0.4437, 0.6918]⟩
$A4$	⟨[0.5286, 0.6929], [0.1518, 0.2920]⟩	⟨[0.4295, 0.5899], [0.2455, 0.3778]⟩	⟨[0.3382, 0.5724], [0.2030, 0.3768]⟩	⟨[0.0000, 0.1110], [0.6467, 0.8536]⟩	⟨[0.0705, 0.2117], [0.4548, 0.7072]⟩
$A5$	⟨[0.6608, 0.8469], [0.0670, 0.1347]⟩	⟨[0.5052, 0.6962], [0.1377, 0.2657]⟩	⟨[0.2586, 0.4519], [0.2869, 0.5080]⟩	⟨[0.1712, 0.3690], [0.3186, 0.5513]⟩	⟨[0.1284, 0.3033], [0.3551, 0.6033]⟩
Positive ideal solutions (A^*)	⟨[0.6608, 0.8469], [0.0670, 0.1347]⟩	⟨[0.5052, 0.6962], [0.1377, 0.2657]⟩	⟨[0.3382, 0.5724], [0.2030, 0.3768]⟩	⟨[0.1712, 0.3690], [0.3186, 0.5513]⟩	⟨[0.1284, 0.3033], [0.3551, 0.6033]⟩
Negative ideal solutions (A^-)	⟨[0.4625, 0.6160], [0.2366, 0.3707]⟩	⟨[0.4295, 0.5899], [0.2455, 0.3778]⟩	⟨[0.2387, 0.4217], [0.3288, 0.5408]⟩	⟨[0.0000, 0.1110], [0.6467, 0.8536]⟩	⟨[0.0705, 0.2117], [0.4548, 0.7072]⟩

$$F^*_{i3} = \begin{bmatrix} \langle [0.0000, 0.0000], [1.0000, 1.0000] \rangle \\ \langle [0.0784, 0.1563], [0.7700, 0.8181] \rangle \\ \langle [0.0523, 0.1042], [0.8209, 0.8787] \rangle \\ \langle [0.1307, 0.2605], [0.6173, 0.6968] \rangle \\ \langle [0.0261, 0.0521], [0.8724, 0.9394] \rangle \end{bmatrix}, \quad F^*_{i4} = \begin{bmatrix} \langle [0.0943, 0.1649], [0.6076, 0.7725] \rangle \\ \langle [0.1155, 0.1993], [0.5701, 0.7323] \rangle \\ \langle [0.1080, 0.1830], [0.5885, 0.7514] \rangle \\ \langle [0.0000, 0.0000], [1.0000, 1.0000] \rangle \\ \langle [0.1712, 0.2902], [0.4926, 0.6459] \rangle, \end{bmatrix}$$

$$F^{*}_{i5}=\begin{bmatrix}\langle[0.0164,0.0389],[0.8879,0.9509]\rangle\\ \langle[0.0534,0.0969],[0.8075,0.8774]\rangle\\ \langle[0.0110,0.0168],[0.9755,0.9782]\rangle\\ \langle[0.0000,0.0000],[1.0000,1.0000]\rangle\\ \langle[0.0623,0.1162],[0.7808,0.8530]\rangle\end{bmatrix}$$

and

$$F^{-}_{i1}=\begin{bmatrix}\langle[0.0887,0.2010],[0.7417,0.7739]\rangle\\ \langle[0.2261,0.4300],[0.4890,0.5329]\rangle\\ \langle[0.3688,0.6014],[0.2830,0.3633]\rangle\\ \langle[0.2804,0.5015],[0.4412,0.4612]\rangle\\ \langle[0.0000,0.0000],[1.0000,1.0000]\rangle\end{bmatrix},\quad F^{-}_{i2}=\begin{bmatrix}\langle[0.1021,0.1886],[0.6592,0.7841]\rangle\\ \langle[0.0510,0.0943],[0.8048,0.8920]\rangle\\ \langle[0.0306,0.0613],[0.8634,0.9298]\rangle\\ \langle[0.0000,0.0000],[1.0000,1.0000]\rangle\\ \langle[0.1327,0.2594],[0.5609,0.7031]\rangle,\end{bmatrix}$$

$$F^{-}_{i3}=\begin{bmatrix}\langle[0.1307,0.2605],[0.6173,0.6968]\rangle\\ \langle[0.0567,0.1235],[0.8017,0.8517]\rangle\\ \langle[0.0827,0.1745],[0.7520,0.7930]\rangle\\ \langle[0.0000,0.0000],[1.0000,1.0000]\rangle\\ \langle[0.1073,0.2198],[0.7076,0.7418]\rangle\end{bmatrix},\quad F^{-}_{i4}=\begin{bmatrix}\langle[0.0849,0.1501],[0.8107,0.8361]\rangle\\ \langle[0.0630,0.1136],[0.8641,0.8820]\rangle\\ \langle[0.0709,0.1312],[0.8371,0.8595]\rangle\\ \langle[0.1712,0.2902],[0.4926,0.6459]\rangle\\ \langle[0.0000,0.0000],[1.0000,1.0000]\rangle,\end{bmatrix}$$

$$F^{-}_{i5}=\begin{bmatrix}\langle[0.0467,0.0804],[0.8794,0.8970]\rangle\\ \langle[0.0094,0.0213],[0.9669,0.9722]\rangle\\ \langle[0.0519,0.1011],[0.8004,0.8720]\rangle\\ \langle[0.0623,0.1162],[0.7808,0.8530]\rangle\\ \langle[0.0000,0.0000],[1.0000,1.0000]\rangle\end{bmatrix}.$$

The values of the $\Re_i$, $\Im_i$, ζ_i, ξ_i are then computed using Eqs. (37)–(48) (Step 12). Consequently, the values of indexes τ_i and η_i are obtained using Eqs. (49) and (50), and the weight values of λ and γ are 0.5 (Step 13). Finally, the *CI* of all operation alternatives is calculated by using Eq. (51) and is given in Table 8 (Step 14). For instance, for the first operation alternative (A_1) we have:

$$\begin{aligned}\Re_1 &= 1-[(0.6926\times0.9659\times1.0000\times0.9057\times0.9836)+(0.4988\times0.9128\times1.0000\times0.8351\times0.9611)]\\ &\quad+\frac{1}{2}[(0.3816\times0.8505\times1.0000\times0.6076\times0.8879)+(0.4694\times0.8967\times1.0000\times0.7725\times0.9509)]\\ &=1-(0.5960+0.3655)+\frac{1}{2}(0.1752+0.3092)=0.2807,\end{aligned}$$

$$\begin{aligned}\Im_1 &= \max\begin{pmatrix}(0.3074+0.5012-1+\frac{1}{2}(0.3816+0.4694)),(0.0341+0.0872-1+\frac{1}{2}(0.8508+0.8967)),\\ (0.0000+0.0000-1+\frac{1}{2}(1.0000+1.0000)),(0.0943+0.1649-1+\frac{1}{2}(0.6076+0.7725)),\\ (0.0164+0.0389-1+\frac{1}{2}(0.8879+0.9509))\end{pmatrix}\\ &=\max(0.2341,-0.0050,0.0000,-0.0507,-0.0254)=0.2341,\end{aligned}$$

$$\begin{aligned}\zeta_1 &= 1-[(0.9113\times0.8979\times0.8693\times0.9151\times0.9533)+(0.7990\times0.8114\times0.7395\times0.8499\times0.9196)]\\ &\quad+\frac{1}{2}[(0.7417\times0.6592\times0.6173\times0.8107\times0.8794)+(0.7739\times0.7841\times0.6968\times0.8361\times0.8970)]\\ &=1-(0.6206+0.3747)+\frac{1}{2}(0.2152+0.3171)=0.2709,\end{aligned}$$

Table 8
Indices values and *CI* by the proposed IVIF-MAGDM model.

Operation alternatives	$\Re_i$	$\Im_i$	ζ_i	ξ_i	Indexes values		*CI*	Final ranking
					τ_i	η_i		
A_1	0.2807	0.2341	0.2709	0.0583	0.6803	0.2699	4.3853	5
A_2	0.2886	0.1153	0.2699	0.1671	0.4752	0.4936	2.5011	4
A_3	0.0815	0.0192	0.4263	0.2934	0.0000	1.0000	1.0000	1
A_4	0.2048	0.0482	0.3399	0.2330	0.2315	0.7396	1.5836	2
A_5	0.4268	0.2934	0.1072	0.0518	1.0000	0.0000	1.7300	3

$$\xi_1 = \max\begin{pmatrix} (0.0887+0.2010-1+\frac{1}{2}(0.7417+0.7739)),(0.1021+0.1886-1+\frac{1}{2}(0.6592+0.7841)), \\ (0.1307+0.2605-1+\frac{1}{2}(0.6173+0.6968)),(0.0849+0.1501-1+\frac{1}{2}(0.8107+0.8361)), \\ (0.0467+0.0804-1+\frac{1}{2}(0.8794+0.8970)) \end{pmatrix}$$
$$= \max(0.0475, 0.0123, 0.0482, 0.0583, 0.0153) = 0.0583,$$

$$\tau_1 = 0.5\left(\frac{0.2807-0.0815}{0.4268-0.0815}\right) + 0.5\left(\frac{0.2341-0.0192}{0.2934-0.0192}\right) = 0.5(0.5769+0.7837) = 0.6803,$$

and

$$\eta_1 = 0.5\left(\frac{0.2709-0.1072}{0.4263-0.1072}\right) + 0.5\left(\frac{0.0583-0.0518}{0.2934-0.0518}\right) = 0.5(0.5130+0.0269) = 0.2699.$$

Thus, $CI_1 = \tau_1 + \frac{1}{\eta_1} = 0.6803 + \frac{1}{0.2699} = 4.3853$.

According to Table 8, the ranking order of five potential alternatives is A_3, A_4, A_5, A_2 and A_1 for the reservoir flood control operation problem. Hence, the best operation alternative is the third alternative (Step 15).

5. Discussion

This section provides the sensitivity analysis on different weights of the majority attributes (i.e., λ and γ) in terms of CI values in the proposed IVIF-MAGDM model as given in Table 9. The computational results in this table reflect the perception that changes in weights of attributes may impact on the outcome of the evaluation somewhat. It is clear that most alternatives keep similar relative rankings vs. different weights of the majority attributes.

The proposed IVIF-MAGDM model based on the compromise ratio method can provide appropriate solutions (i.e., different order ranking) by considering different value of λ and γ depending on specific applications in the reservoir flood control operation. An effective selection procedure is essential to enhance the decision quality for the experts or DMs. In this paper, the group decision making process is investigated and a multi-attributes framework is introduced for ranking and selecting the best operation alterative. The computational results demonstrate that the proposed model takes into account the requirements as well as the weights of attributes and assesses the alternatives according to these requirements for the engineering application under the IVIF environment. The model assesses and selects the alternatives in terms of the ideal reference points by a group of experts; consequently, the alternatives are ranked in reference to their performance ratings against each other. Thus, the proposed IVIF-MAGDM based on the compromise ratio method is a suitable approach within the multi-attributes analysis for the real-life situations.

Table 9
Different values of λ and γ and preference order ranking by the proposed IVIF-MAGDM model.

λ and γ Values	Alternatives	CI	Preference order ranking
$\lambda = 0$ and $\gamma = 0$	A_1	2.7332	5
	A_2	2.3115	4
	A_3	1.0000	1
	A_4	1.4772	2
	A_5	1.8506	3
$A = 0.2$ and $\gamma = 0.2$	A_1	3.1478	5
	A_2	2.3868	4
	A_3	1.0000	1
	A_4	1.5197	2
	A_5	1.8099	3
$A = 0.5$ and $\gamma = 0.5$	A_1	4.3853	5
	A_2	2.5011	4
	A_3	1.0000	1
	A_4	1.5836	2
	A_5	1.7300	3
$A = 0.8$ and $\gamma = 0.8$	A_1	8.6760	5
	A_2	2.6169	4
	A_3	1.0000	1
	A_4	1.6477	3
	A_5	1.6057	2
$A = 1$ and $\gamma = 1$	A_1	37.7477	5
	A_2	2.6951	4
	A_3	1.0000	1
	A_4	1.6905	3
	A_5	1.4195	2

Following are the main important aspects of the proposed IVIF-MAGDM model based on the compromise ratio method against the identified literature: (1) a modern fuzzy set in an IVIF-form for the compromise ratio approach in a multi-attributes framework is taken into consideration to demonstrate more flexibility and better representation uncertainties than traditional fuzzy sets for complex decision problems; (2) a new IVIF-relative closeness based on two IVIF-indices and the score and accuracy functions is introduced that effectively considers the relative distance of potential alternatives from the IVIF-positive-ideal and IVIF-negative-ideal solutions; and (3) two operations on IVIFSs, namely subtraction and division, are developed according to the deconvolution for equations by the addition and multiplication operations.

6. Concluding remarks

This paper presented a novel multi-attribute group decision making (MAGDM) approach in an interval-valued intuitionistic fuzzy (IVIF) environment. Two new operations for interval-valued intuitionistic fuzzy sets were developed by taken the operational laws of interval-valued intuitionistic fuzzy numbers (IVIFN) into consideration. The proposed IVIF-MAGDM model was based on the concept of the compromise ratio method and modern fuzzy sets under the group decision making process. Experts or decision makers described linguistic terms to evaluate the importance of the selected attributes and to assess the each alternative vs. each attribute. These linguistic terms were converted into IVIFNs and IVIF-decision matrix was formed. The interval-valued intuitionistic fuzzy weighted geometric averaging (IVIFWGA) operator has been utilized to aggregate judgments of the experts. After IVIF-positive-ideal and IVIF-negative-ideal points were defined, the IVIF-relative distance of each alternative from these points was calculated by constructing the IVIF-positive-ideal separation and IVIF-negative-ideal separation matrices. Finally, an extended collective index in the IVIF environment has been presented based on the score and accuracy functions of two IVIF-indices to simultaneously take into account not only the shortest distance of each candidate from the positive-ideal point but also the farthest distance from the negative-ideal point. The presented model also avoided the difficulties from the extension of the traditional compromise solution methods in the IVIF-environment. An application example from the recent literature for the engineering application to reservoir flood control operation was presented to exemplify the proposed IVIF-MAGDM model in detail. The underlying concepts employed in this paper were intelligible to the group decision making process under uncertainty. The needed computations are straightforward and easy-to-use in real-life applications. Also, it assists managers in making critical decisions during the selection of the best alternatives for solving complex decision making problems. For future research, extending other MAGDM methods, such as SAW, PROMETHEE and ELECTRE, is recommended under uncertainty. In addition, the data consistency in the experts' information can be regarded through the group decision making process under the IVIF-environment.

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