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A q-NUMERICAL RADIUS INEQUALITY AND CHARACTERIZATION OF SOME TYPES OF MATRICES

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Abstract. We discuss about characterization of square matrices $A \in M_n$ that satisfy the inequality $r_q(A) \leq \frac{r(A)(1+q)}{2}$, for every $0 \leq q \leq 1$. Also we show that this inequality is equivalent to the inequality $\rho(AB) \leq r(A)r(B)$, for every $B \in M_n$, with $\text{rank}(B) = 1$. Here $r_q(A)$, $r(A)$ and $\rho(A)$, denote the q-numerical radius, ordinary numerical radius and spectral radius of A respectively.

1. Introduction

The study of numerical range of operators on Banach or Hilbert spaces represents one of the active research areas in operator theory [2, 4]. Also inequalities about numerical radius is an important topic in the field of numerical range. A new good reference which has gathered various types of these inequalities is [3]. One of the interesting questions is characterizing all matrices $A \in M_n$ that satisfy the inequity $\rho(AB) \leq r(A)r(B)$, for every $B \in M_n$. It is proved that this inequality implies that A is a radial matrix which is unitarily similar to a matrix of the form $\|A\|(I_k \oplus B)$ and the numerical range of B is a subset of $\{z : |z - \frac{1}{2}| \leq \frac{1}{2}\}$ [1]. The converse is not true and it is a conjecture that A is unitarily similar to a matrix of the form $\|A\|(I_k \oplus 0_r \oplus B)$ with

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$\mathrm{Re}(W(B^{-1})) \geq 1$ (B is omitted if $k+r=n$). Another question that arises from this inequality is about equivalency of it with the inequality $\rho(AB) \leq r(A)r(B)$, for every $B \in M_n$, with $\mathrm{rank}(B)=1$. In this paper we get an equivalent inequality in the form of q-numerical range for the last inequality and discuss about the matrices which satisfy in it.

2. Main results

In this section e_i denotes the i-th vector of the standard basis in $\mathbb{C}^n$. For a matrix $A \in M_n$, the diagonal matrix whose diagonal is equal to diagonal of A is displayed by $\mathrm{diag}(A)$. Also the Euclidean norm on $\mathbb{C}^n$ is displayed by $|.|$. We remember that the numerical range of $A \in M_n$ is defined by

$$W(A) = \{x^*Ax : x^*x = 1\},$$

and for $0 \leq q \leq 1$, the q-numerical range of A is as follow [5]:

$$W_q(A) = \{y^*Ax : y^*y = x^*x = 1,\ y^*x = q\}.$$

It is clear that $W_1 = W$. The numerical radius $r(.)$ and q-numerical radius $r_q(.)$ are defined as follow:

$$r(A) = \sup\{|\lambda| : \lambda \in W(A)\},$$

and

$$r_q(A) = \sup\{|\lambda| : \lambda \in W_q(A)\}.$$

Also Crawford number of A is defied by

$$c(A) = \inf\{|\lambda| : \lambda \in W(A)\},$$

Lemma 2.1. *Let A and $\tilde{A}$ be two unitarily similar matrices in M_n. Then $\rho(AB) \leq r(A)r(B)$, for every $B \in M_n$ with rank$(B)=1$, if and only if $\rho(\tilde{A}B) \leq r(\tilde{A})r(B)$, for every $B \in M_n$ with rank$(B)=1$,*

Proof. Suppose that U is a unitary matrix in M_n such that $\tilde{A} = U^*AU$ and $\rho(AB) \leq r(A)r(B)$, for every $B \in M_n$ with $\mathrm{rank}(B)=1$. Now for every $B \in M_n$ with $\mathrm{rank}(B)=1$, we have

$$\rho(\tilde{A}B) = \rho(U^*AUBU^*U) = \rho(AUBU^*) \leq r(A)r(UBU^*) = r(\tilde{A})r(B).$$

□

Theorem 2.2. *For $A \in M_n$, two following conditions are equivalent:*
(i) $\rho(AB) \leq r(A)r(B)$, for every matrix B with rank$(B)=1$. (2.1)
(ii) $r_q(A) \leq \frac{r(A)(1+q)}{2}$, for every $0 \leq q \leq 1$.

Proof. Let B be a rank one matrix in M_n with $\|B\| = 1$. Then there exist unit vectors $x, y \in \mathbb{C}^n$ such that $B = xy^*$. Now choose a unitary matrix U in such a way that its first row is equal to x^* and its second row is $\frac{y^*-(y^*x)x^*}{\sqrt{1-|y^*x|^2}}$. Then $Ux = e_1$ and $Uy = x^*ye_1 + \sqrt{1-|y^*x|^2}e_2$. Therefore

$$xy^* = U^*(e_1(y^*xe_1^* + \sqrt{1-|y^*x|^2}e_2^*))U = U^*(y^*xE_{11} + \sqrt{1-|y^*x|^2}E_{22})U.$$

Hence

$$r(B) = \frac{1+|y^*x|}{2}.$$

On the other hand we have

$$\rho(AB) = \rho(Axy^*) = \rho(y^*Ax) = |y^*Ax|.$$

Therefore (i) holds if and only if (ii) holds.

□

Corollary 2.3. *Let $A = (a_{ij})$ be a matrix in M_n that satisfies the condition (2.1). Then*

$$max\{|a_{ij}| \ : \ i \neq j\} \leq \frac{r(A)}{2}.$$

Proof. For $i \neq j$, setting $y = e_i$ and $x = e_j$, by Theorem 2.2 we have,

$$|a_{ij}| = |e_i^*Ae_j| \leq r_0(A) \leq \frac{r(A)}{2}.$$

□

Corollary 2.4. *Let A be a matrix in M_n that satisfies the condition (2.1). Then*
$\|A - diag(A)\|_1, \|A - diag(A)\|_\infty \leq \frac{r(A)\sqrt{n-1}}{2}$.

Proof. Let $a_{rs} = |a_{rs}|\exp(i\theta_{rs})$, $y = e_r$ and $x = \frac{\sum_{r\neq s}^n \exp(-i\theta_{rs})e_r}{\sqrt{n-1}}$ and using Theorem 2.2, for every $1 \leq r \leq n$, we have,

$$\frac{\sum_{s\neq r}^n |a_{rs}|}{\sqrt{n-1}} = |y^*Ax| \leq r_0(A) \leq \frac{r(A)}{2}.$$

Hence $\|A - \mathrm{diag}(A)\|_1 \leq \frac{\|A\|\sqrt{n-1}}{2}$. Similarly we can show that $\|A - \mathrm{diag}(A)\|_\infty \leq \frac{r(A)\sqrt{n-1}}{2}$.

□

Corollary 2.5. *Let A be a matrix in M_n that satisfies the condition (2.1). Then*

$$W(A) \subseteq D(tr(\frac{A}{n}), \frac{r(A)\sqrt{n-1}}{2}).$$

Proof. By [5, Theorem 1.3.4], A is unitarily similar to a matrix B whose diagonal entries are equal to $\mathrm{tr}(\frac{A}{n})$. Now considering [5, Theorem 1.5.2], a discussion same as Corollary 2.3, gets the desired result.

□

Lemma 2.6. *Let A be a matrix in M_n that satisfies the condition (2.1). Then A is a radial matrix.*

Proof. We can take unit vectors $x, y \in \mathbb{C}^n$ in such a way that $\|A\| = |y^*Ax|$ and $y^*x \geq 0$. Since xy^* is a rank one matrix, by theorem 2.2, we have:

$$\|A\| = |y^*Ax| \leq \frac{r(A)(1+y^*x)}{2} \leq r(A).$$

Therefore A is a radial matrix.

□

Lemma 2.7. *Let A be an invertible matrix in M_n that satisfies the condition (2.1). Then*

(i) $\frac{1}{\|A^{-1}\|} \leq [\frac{1+c(A^{-1})}{2}]r(A)$.

(ii) $inf\,\{|Ax| \ : \ |x| = 1\} \geq \frac{2}{r(A)[1+r(A^{-1})]}$.

Proof. For every unit vector x in $\mathbb{C}^n$ we have

$$\frac{1}{|A^{-1}x|} = \frac{|x^*A(A^{-1}x)|}{|A^{-1}x|} = \rho(\frac{A(A^{-1}x)x^*)}{|A^{-1}x|}) \leq r(A)\frac{r(A^{-1}xx^*)}{|A^{-1}x|} = r(A)\frac{1+x^*A^{-1}x}{2}.$$

Now taking the infimum (supremum) over x gets (i)((ii)).

□

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