A $q$-NUMERICAL RADIUS INEQUALITY AND CHARACTERIZATION OF SOME TYPES OF MATRICES

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Abstract. We discuss about characterization of square matrices $A \in M_n$ that satisfy the inequality $r_q(A) \leq r(A)(1+\frac{q}{2})$, for every $0 \leq q \leq 1$. Also we show that this inequality is equivalent to the inequality $\rho(AB) \leq r(A)r(B)$, for every $B \in M_n$, with rank($B$) = 1. Here $r_q(A)$, $r(A)$ and $\rho(A)$, denote the $q$-numerical radius, ordinary numerical radius and spectral radius of $A$ respectively.

1. Introduction

The study of numerical range of operators on Banach or Hilbert spaces represents one of the active research areas in operator theory [2, 4]. Also inequalities about numerical radius is an important topic in the field of numerical range. A new good reference which has gathered various types of these inequalities is [3]. One of the interesting questions is characterizing all matrices $A \in M_n$ that satisfy the inequity $\rho(AB) \leq r(A)r(B)$, for every $B \in M_n$. It is proved that this inequality implies that $A$ is a radial matrix which is unitarily similar to a matrix of the form $\|A\| (I_k \oplus B)$ and the numerical range of $B$ is a subset of $\{z : |z - \frac{1}{2}| \leq \frac{1}{2}\}$ [1]. The converse is not true and it is a conjecture that $A$ is unitarily similar to a matrix of the form $\|A\| (I_k \oplus 0, \oplus B)$ with

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Re($W(B^{-1})$) $\geq 1$ ($B$ is omitted if $k + r = n$). Another question that arises from this inequality is about equivalency of it with the inequality $\rho(AB) \leq r(A)r(B)$, for every $B \in M_n$, with rank($B$) = 1. In this paper we get an equivalent inequality in the form of $q$-numerical range for the last inequality and discuss about the matrices which satisfy in it.

2. Main results

In this section $e_i$ denotes the $i$-th vector of the standard basis in $\mathbb{C}^n$. For a matrix $A \in M_n$, the diagonal matrix whose diagonal is equal to diagonal of $A$ is displayed by diag($A$). Also the Euclidean norm on $\mathbb{C}^n$ is displayed by $|.|$. We remember that the numerical range of $A \in M_n$ is defined by

$$W(A) = \{x^*Ax : x^*x = 1\},$$

and for $0 \leq q \leq 1$, the $q$-numerical range of $A$ is as follow [5]:

$$W_q(A) = \{y^*Ax : y^*y = x^*x = 1, \ y^*x = q\}.$$  

It is clear that $W_1 = W$. The numerical radius $r(\cdot)$ and $q$-numerical radius $r_q(\cdot)$ are defined as follow:

$$r(A) = \sup\{||\lambda|| : \lambda \in W(A)\},$$

and

$$r_q(A) = \sup\{||\lambda|| : \lambda \in W_q(A)\}.$$  

Also Crawford number of $A$ is defied by

$$c(A) = \inf\{||\lambda|| : \lambda \in W(A)\},$$

Lemma 2.1. Let $A$ and $\tilde{A}$ be two unitarily similar matrices in $M_n$. Then $\rho(AB) \leq r(A)r(B)$, for every $B \in M_n$ with rank($B$) = 1, if and only if $\rho(\tilde{A}B) \leq r(\tilde{A})r(B)$, for every $B \in M_n$ with rank($B$) = 1,

Proof. Suppose that $U$ is a unitary matrix in $M_n$ such that $\tilde{A} = U^*AU$ and $\rho(AB) \leq r(A)r(B)$, for every $B \in M_n$ with rank($B$) = 1. Now for every $B \in M_n$ with rank($B$) = 1, we have

$$\rho(\tilde{A}B) = \rho(U^*AUU^*U) = \rho(AU^*BU^*) \leq r(A)r(UB^*) = r(\tilde{A})r(B).$$

Theorem 2.2. For $A \in M_n$, two following conditions are equivalent:

(i) $\rho(AB) \leq r(A)r(B)$, for every matrix $B$ with rank($B$) = 1. (2.1)

(ii) $r_q(A) \leq \frac{r(A)(1+q)}{2}$, for every $0 \leq q \leq 1$. 


Proof. Let $B$ be a rank one matrix in $M_n$ with $\|B\| = 1$. Then there exist unit vectors $x, y \in \mathbb{C}^n$ such that $B = xy^*$. Now choose a unitary matrix $U$ in such a way that its first row is equal to $x^*$ and its second row is $y^* - (y^*x)x^*/\sqrt{1 - |y^*x|^2}$. Then $UX = e_1$ and $UY = x^*y_1 + \sqrt{1 - |y^*x|^2}e_2$.

Therefore

$$xy^* = U^*(e_1(y^*xe_1^* + \sqrt{1 - |y^*x|^2}e_2^*))U = U^*(y^*xe_{11} + \sqrt{1 - |y^*x|^2}e_{22})U.$$ 

Hence

$$r(B) = 1 + |y^*x|/2.$$ 

On the other hand we have

$$\rho(AB) = \rho(Axy^*) = \rho(y^*Ax) = |y^*Ax|.$$ 

Therefore (i) holds if and only if (ii) holds.

\[ \square \]

Corollary 2.3. Let $A = (a_{ij})$ be a matrix in $M_n$ that satisfies the condition (2.1). Then

$$\max\{|a_{ij}| : i \neq j\} \leq \frac{r(A)}{2}.$$ 

Proof. For $i \neq j$, setting $y = e_i$ and $x = e_j$, by Theorem 2.2 we have,

$$|a_{ij}| = |e_i^*Ae_j| \leq r_0(A) \leq \frac{r(A)}{2}.$$ 

\[ \square \]

Corollary 2.4. Let $A$ be a matrix in $M_n$ that satisfies the condition (2.1). Then

$$\|A - \text{diag}(A)\|_1, \|A - \text{diag}(A)\|_\infty \leq \frac{r(A)\sqrt{n-1}}{2}.$$ 

Proof. Let $a_{rs} = |a_{rs}|\exp(i\theta_{rs})$, $y = e_r$ and $x = \sum_{s \neq r}^n \exp(-i\theta_{rs})e_r/\sqrt{n-1}$ and using Theorem 2.2, for every $1 \leq r \leq n$, we have,

$$\frac{\sum_{s \neq r}^n |a_{rs}|}{\sqrt{n-1}} = |y^*Ax| \leq r_0(A) \leq \frac{r(A)}{2}.$$ 

Hence $\|A - \text{diag}(A)\|_1 \leq \frac{\|A\|\sqrt{n-1}}{2}$. Similarly we can show that $\|A - \text{diag}(A)\|_\infty \leq \frac{r(A)\sqrt{n-1}}{2}$. 

\[ \square \]
Corollary 2.5. Let $A$ be a matrix in $M_n$ that satisfies the condition (2.1). Then

$$W(A) \subseteq D \left( \frac{\text{tr}(A)}{n}, \frac{r(A)\sqrt{n-1}}{2} \right).$$

Proof. By [5, Theorem 1.3.4], $A$ is unitarily similar to a matrix $B$ whose diagonal entries are equal to $\text{tr}(\frac{A}{n})$. Now considering [5, Theorem 1.5.2], a discussion same as Corollary 2.3, gets the desired result.

Lemma 2.6. Let $A$ be a matrix in $M_n$ that satisfies the condition (2.1). Then $A$ is a radial matrix.

Proof. We can take unit vectors $x, y \in \mathbb{C}^n$ in such a way that $\|A\| = |y^*Ax|$ and $y^*x \geq 0$. Since $xy^*$ is a rank one matrix, by theorem 2.2, we have:

$$\|A\| = |y^*Ax| \leq \frac{r(A)(1 + y^*x)}{2} \leq r(A).$$

Therefore $A$ is a radial matrix.

Lemma 2.7. Let $A$ be an invertible matrix in $M_n$ that satisfies the condition (2.1). Then

(i) $\frac{1}{\|A^{-1}\|} \leq \frac{1 + c(A^{-1})}{2} r(A)$.

(ii) $\inf \{ |Ax| : |x| = 1 \} \geq \frac{2}{r(A)[1 + r(A^{-1})]}.$

Proof. For every unit vector $x$ in $\mathbb{C}^n$ we have

$$\frac{1}{|A^{-1}x|} = \left| \frac{x^*A(A^{-1}x)}{|A^{-1}x|} \right| = \frac{\rho(A^{-1}x)}{|A^{-1}x|} \leq r(A) \frac{r(A^{-1}xx^*)}{|A^{-1}x|} = r(A) \frac{1 + x^*A^{-1}x}{2}.$$

Now taking the infimum (supremum) over $x$ gets (i)(ii).

References