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Finite dimensional Banach spaces with the same normaloid and radial operators

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Abstract: We characterize all finite dimensional Banach spaces that every normaloid operator on them is radial.

Keywords: Banach space, spectral radius, numerical radius.

1 INTRODUCTION

Let X be a finite dimensional real or complex Banach space, $S(X)$ be its unit sphere and X^* denotes its dual. A pair $(x, f) \in S(X) \times S(X^*)$ is a dual pair if $f(x) = 1$. For every $x \in S(X)$ and $f \in S(X^*)$ we define

$$D(x) = \{g : (x, g) \text{ is a dual pair}\},$$

$$D(f) = \{y : (y, f) \text{ is a dual pair}\}.$$

By Hahn-Banach theorem, these sets are not empty. The elements of $D(x)$ are called support functionals of x. If $D(x)$ has more than one element, it is said that $S(X)$ is not smooth at x, otherwise it will be smooth at x. Also if $D(f)$ has more than one element, then it is called a segment. Note that if a segment pass through x and y and $0 \leq \lambda \leq 1$, then $\lambda x + (1-\lambda)y$ locates on that segment. A norm is called strictly convex if it contains no segment and it is smooth if $S(X)$ is smooth at every point. It is a well known that $S(X)$ is smooth (strictly convex) if and only if $S(X^*)$ is strictly convex (smooth).

For every T in $B(X)$, the algebra of bounded linear operators on X, the Bauer or spacial numerical range is defined as follow [1, 2]

$$W(T) = \{f(Tx) : (x, f) \text{ is a dual pair}\}.$$

The numerical radius of T is defined by

$$r(T) = \sup\{|\lambda| : \lambda \in W(T)\}.$$

It is clear that $\rho(T) \leq r(T) \leq \|T\|$, where $\|.\|$ and $\rho(.)$ denote the operator norm and spectral radius respectively. An operator $T \in B(X)$ is called normaloid, if $r(T) = \|T\|$ and it is radial if $\rho(T) = \|T\|$. It is obvious that every radial operator is normaloid. In the case of X be a Hilbert space, Wintner [4] proved that the converse of this fact is also true. In addition Lumer [3] proved that if X is a (not necessarily finite dimensional) uniformly convex Banach space, then every normaloid operator is radial. Since in the finite dimensional case uniform and strict convexity are the same, if T^* is the adjoint of T, the equalities $\|T\| = \|T^*\|$, $r(T) = r(T^*)$, $\rho(T) = \rho(T^*)$ and Lumer theorem imply that if $S(X^*)$ is strictly convex or equivalently $S(X)$ is smooth, then again every normaloid operator will be radial. The main purpose of this paper is finding a necessary and sufficient condition on X that every normaloid operator on X be radial. In Theorem 2.1 we will give this condition. For example, this theorem gets a clear characterization of the norms on $\mathbb{R}^2$ that every normaloid operator on them is radial. They are those norms that sharp points of their unit sphere are not located at the end points of flat portions of the unit sphere.

2 Main result

Before stating the main theorem of this paper, we remember that every pair (x, f) in $X \times X^*$ defines a rank one operator $T = x \otimes f$ on X, that is introduced by $T(.) = f(.)x$.

Theorem 2.1. Let X be a finite dimensional Banach space. Then the following expressions are equivalent:

(i) Every normaloid operator on X is radial.

(ii) Every rank one normaloid operator on X is radial.

(iii) Every two points on a segment of $S(X)$, have the same support functionals.

Proof. Clearly we have $(i)->(ii)$.

$(ii)->(iii)$ Suppose (iii) dose not satisfy. Then there exist $x, y \in S(X)$ and functionals $f, h \in S(X^*)$ such that $f(x) = f(y) = h(x) = 1$ and $h(y) \neq 1$. Now consider the family of functionals

$$\{g(\lambda) = \lambda f + (1-\lambda)h : \ 0 < \lambda < 1\}.$$

Clearly $(x, g(\lambda))$ is a dual pair, for every $0 < \lambda < 1$. If $|g(\lambda)(y)| = 1$, for every $0 < \lambda < 1$, then we will have $h(y) = 1$, which is a contradiction. Hence there exists $g = g(\lambda_0)$ such that $|g(y)| < 1$. Setting $T = y \otimes g$, we have $\|T\| = 1$. Since (x, f) is a dual pair and $f(T(x)) = g(x)f(y) = 1$, then $r(T) = 1$. Also it is easy to see that $\rho(T) = |g(y)| < 1$. Hence T is normaloid but not radial.

$(iii)->(i)$ Suppose (iii) is satisfied and consider T as a normaloid operator on X. Without loss of generality we can suppose that $\|T\| = 1$ and there is a dual pair (x, f) such that $f(Tx) = 1$. If $Tx = x$ or $T^*f = f$, then T will be radial and there is nothing to proof. Suppose that $Tx \neq x$ and $T^*f \neq f$. Then (x, f), (Tx, f) and (x, T^*f) are distinct dual pairs. Since both x and Tx are in the segment $D(f)$, we should have $D(x) = D(T(x))$, and so $(T^*f)(Tx) = 1$. Therefore (T^2x, f) is a dual pair and $\|T^2\| = 1$. Now if $(T^2)^*f = f$, then $\rho(T)^2 = \rho(T^2) = \rho((T^2)^*) = 1$, and T will be radial. Otherwise, we have $(T^2)^*f \neq f$ and so (x, f), (Tx, f) and $(x, (T^2)^*f)$ are distinct dual pairs. A same argument shows that (T^3x, f) is a dual pair and $\|T^3\| = 1$. In general, at step k, we should have $\|T^k\| = 1$, otherwise $x, Tx \in D(f)$ and $(T^k)^*f \in D(x) - D(Tx)$ which contradicts (iii). Therefore $\|T^n\| = 1$, for all positive integer n and so $\rho(T) = 1$. Hence T will be radial. □

Note that the proof of part $(ii)->(iii)$ of the above theorem, can be applied for infinite dimensional Banach spaces. But in the proof of $(iii)->(i)$, we use of closeness of the numerical range which is a conclusion of finite dimensionality of X.

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