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ORIGINAL ARTICLE

Identifying the time of a step change in bivariate binomial processes

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Abstract Control charts are one of the most applicable tools in statistical process control. The time in which the control chart signals an out-of-control alarm is not the actual time in which the change has occurred. In other words, control chart detects the change with some delay. The actual time of the change taking place is referred to as change point. Change point estimation facilitates the identification of cause(s) of change and reduces the corresponding time and cost. There are many processes in which the control of two correlated attributes is necessary. Multiattribute control charts are used in such cases due to correlation between attributes. In this paper, two methods including maximum likelihood estimation (MLE) and clustering are proposed for estimating change point in nonconformity ratio vector of a process with bivariate binomial distribution. Also, the performances of these methods are evaluated and compared by Monte Carlo simulations.

Keywords Change point · Bivariate binomial · Maximum likelihood function · Clustering method · Attribute control charts · Statistical process control

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1 Introduction

One of the main objectives of organizations is producing high-quality products and services in order to retain their competitiveness. This purpose is realized through monitoring and improving production or service processes. Control charts are effective tools for monitoring and signaling out-of-control processes.

As a change is detected, searching for the cause commences. Usually, there is a lag between the actual time of change taking place and the time the change is signaled by the control chart. This lag highly depends on the size of the shift. The actual time of the change in a process is called change point. Finding the exact time of change leads to easier detection of the assignable cause through limiting the period within the change has happened. Change point estimation also reduces the process control costs by shortening the time spent of finding the root cause of problems.

Different approaches are introduced by researchers to estimate change point under different assumptions. One of the most prevalent approaches is using maximum likelihood estimation. For example, Samuel et al. [1] introduced an approach based on maximum likelihood function for estimating the change point in the mean of normal processes while being monitored by $\overline{X}$ Shewhart control chart. Samuel et al. [2] developed a change point estimator using the same approach for the variance of normal processes in S or R control charts. Pignatiello and Samuel [3] proposed a maximum likelihood estimator to find the real time of a change in nonconforming fraction of a process.

Some other approaches take advantage of genetic algorithm, clustering and artificial neural network. For example, Jann [4] developed an estimator for multiple changes using genetic algorithm or Ghazanfari et al. [5] applied clustering method for change point estimation in Shewhart control charts, and Atashgar and Noorossana [6] used artificial neural network for this purpose.

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On the other hand, different types of change, such as single and multiple step shifts, drift, and monotonic, are considered in the literature; Perry and Pignatiello [7] proposed a change point estimator for $\overline{X}$ control chart when the process changes with a linear trend while Noorossana et al. [8] developed an estimator for the same control chart considering monotonic changes.

There are many processes in which more than one quality characteristic should be monitored simultaneously to determine whether the process is in control or not. Hence, several estimators are proposed in the literature for multivariate cases.

Nedumaran and Pigmatiello [9] developed maximum likelihood estimator to identify the real time of sudden step changes in mean of multivariate normal processes when the process is monitored by chi-square control chart. Zamba and Hawkins [10] used likelihood ratio test to estimate change point in the mean vector of multivariate normal processes while being monitored by T^2 control chart. Also, Ahmadzadeh [11] and Noorossana et al. [12] used artificial neural network for change point estimation in multivariate control charts. For more information about change types, estimation approaches, and future researches related to change point estimation, refer to Amiri and Allahyari [13].

Sometimes, more than one attribute describes the quality of a process. In these cases, the multiattribute control charts are used to monitor the process. Finding the real time of a change in these control charts is vital and helps quality engineers to find the root causes of the problem in the process as quickly as possible. Change point estimation in multiattribute control charts is considered by some researchers. Niaki and Khedmati [14, 15] proposed maximum likelihood estimation (MLE) for finding the real time of step and linear trend changes in multivariate Poisson processes, respectively. Niaki and Khedmati [16] also considered monotonic change point estimation in multivariate Poisson processes. Afterwards, Niaki and Khedmati [17] proposed two control charts to monitor multivariate binomial processes as well as a MLE estimator for the change point estimation. Moreover, Niaki and Khedmati [18] considered change point estimation in multivariate binomial process under both step change and drift recently. They first transformed multivariate binomial distribution to multivariate normal distribution. Then, they derived the MLE for change pint estimation based on the multivariate normal distribution under both step change and drift. They also get a signal from a T^2 hotelling control chart after transforming multiattribute binomial distribution to multivariate normal distribution.

In this paper, we consider a bivariate binomial process, and without any transformation, we monitor the process with a multivariate np (Mnp) control chart discussed in details in the next section. In addition, we develop two change point estimators using MLE and clustering approaches to find the real time of a step change. Note that the MLE is developed based on the bivariate binomial distribution, and no transformation is used in this paper. This is the main difference between the MLE developed by Niaki and Khedmati [17, 18] and the one developed in this paper. To the best of our knowledge, the clustering approach in estimating change point of a bivariate binomial process is not considered yet.

The rest of the paper is organized as follows: Mnp control chart for monitoring bivariate binomial processes is discussed in Section 2. The proposed methods are illustrated in Section 3. Assessment and comparison of these methods are done in Section 4, and finally, concluding remarks are given in Section 5.

2 Mnp control chart

In practice, we face many processes in which monitoring of more than one attribute quality characteristic is required to determine if the process is in control or not. Consider a casting process in which the amounts of cracks and holes on the surface are monitored over time.

Several control charts are designed for monitoring multiattribute processes. For example, Patel [19] proposed a control chart based on T^2 values. Marcucci [20] used multinomial distribution to classify conforming and nonconforming products.

Lu et al. [21] proposed a control chart called multivariate np (Mnp) by weighted average and based on univariate Shewhart control charts in which the correlation between attribute quality characteristics is considered. They proposed the control charts for m attributes. However, since we consider two attributes in this paper, the statistic and control limits are explained based on two attributes. Assume that $\mathbf{p}=(p_1, p_2)$ is the ratio vector (probabilities of facing the first and second defects, respectively) and $\sum$ is a 2 × 2 covariance matrix between two attributes. The probabilities vector of $\mathbf{p}$ and covariance matrix of $\sum$ are assumed to be known and estimated from phase I studies. Based on the sampled observations, the statistic of this control chart named as X is described as follows:

$$X = \sum_{j=1}^{2} c_j / \sqrt{p_j} . \quad (1)$$

In which c_j indicates the number of products containing the ith defect, p_j shows the probability of facing the jth defect, and

2 is the number of defects. The mean and variance of the statistic X can be written as follows:

$$E(X) = \sum_{j=1}^{2} np_j/\sqrt{p_j} = n\sum_{j=1}^{2}\sqrt{p_j}, \quad (2)$$

$$\mathrm{Var}(X) = n\left(\sum_{j=1}^{2}\left(1-p_j\right) + 2\sigma_{12}\sqrt{(1-p_1)(1-p_2)}\right), \quad (3)$$

where n is number of products sampled at each time and σ_{12} is the covariance between attributes.

The center line and control limits are described as

$$\mathrm{UCL} = n\sum_{j=1}^{2}\sqrt{p_j} + 3\sigma_X, \quad (4)$$

$$\mathrm{CL} = n\sum_{j=1}^{2}\sqrt{p_j}, \quad (5)$$

$$\mathrm{LCL} = n\sum_{j=1}^{2}\sqrt{p_j} - 3\sigma_X. \quad (6)$$

σ_X is the standard deviation of the statistic X, and it is square root of Var(X) in Eq. (3). Also, to prevent from negative LCL, the number of products sampled (n) is defined such that constraint $n \geq \frac{3m}{\sum_{j=1}^{m} p_j}$ is satisfied. When X falls beyond the control limits, the control chart gives an out-of-control signal. As mentioned, the time of out-of-control signal is not the real time of the change, so the change point should be estimated to speed up the correction process. Two change point estimators are proposed in Section 3.

3 Proposed methods

Assume a bivariate binomial processes which is being monitored by a Mnp control chart. Two methods are proposed for estimating change point in this process.

3.1 Change point estimation using maximum likelihood function

In the MLE approach, the change point is the time in which the likelihood function is maximized. The estimated change point is given as

$$\widehat{\tau} = \arg\max\{L(t)|\ 0 \leq t < T\}, \quad (7)$$

where $L(t)$ is the likelihood function including both in control and out-of-control observations, t is the time samples are taken (when the first sample is taken, t is set equal to 0 and when the control chart signals the t is equal to T), and $\widehat{\tau}$ is the maximum likelihood change point estimator.

3.1.1 Bivariate binomial distribution

In many cases, monitoring two correlated attribute characteristics (x, y) which follows bivariate binomial distribution is needed. Probability function of bivariate binomial distribution is defined as follows:

$$g(x,y) = \sum_{n_{11}} \frac{n!}{n_{11}!(x-n_{11})!(y-n_{11})!(n-x-y+n_{11})!} \times p_{11}^{n_{11}}(p_x-p_{11})^{x-n_{11}}\left(p_y-p_{11}\right)^{y-n_{11}}\left(1-p_y-p_x+p_{11}\right)^{n-x-y+n_{11}}. \quad (8)$$

In Eq. (8), we have $x = n_{10} + n_{11}, y = n_{01} + n_{11}, p_x = p_{11} + p_{10}$, $p_y = p_{11} + p_{01}$ and $\max\{0, x + y - n\} \leq n_{11} \leq \min\{x,y\}$. Also

$$\rho_{\mathrm{x,y}} = \frac{p_{11} - p_{\mathrm{x}}p_{\mathrm{y}}}{\sqrt{p_{\mathrm{x}}(1-p_{\mathrm{x}})p_{\mathrm{y}}\left(1-p_{\mathrm{y}}\right)}}, \quad (9)$$

is the correlation between x and y attribute quality characteristics. All parameters in Eq. (8) are defined in Table 1.

Table 1 Possible scenarios of defects, their probability, and numbers

Scenarios	Number	Probability
Only defect 1	n_{10}	p_{10}
Only defect 2	n_{01}	p_{01}
Both defects 1 and 2	n_{11}	p_{11}
Defect 1	x	p_x
Defect 2	y	p_y

When the Mnp control chart signals an out-of-control process at $t = T$, it means that the process parameters are changed in an unknown point of $t = \tau$. From $t = 1, \ldots, \tau$ the nonconforming ratio vector changes and the process continues from $t = \tau + 1$ with the new ratio vector. We use the third subscript for the parameters in the likelihood function to distinguish between the in-control and out-of-control process parameters. Subscript 0 shows the in-control parameters and the subscript 1 is used for the out-of-control parameters. Based on the third subscript as well as the change point at time τ, the likelihood function can be written as below using the joint probability mass function of the bivariate binomial distribution given in Eq. (8):

$$\begin{aligned} L(t,X) &= \prod_{t=1}^{\tau} g_{in}(x,y) \times \prod_{t=\tau+1}^{T} g_{out}(x,y) = \prod_{t=1}^{\tau}\sum_{n_{11}}\Bigg(\frac{n!}{n_{11}!(x-n_{11})!(y-n_{11})!(n-x-y+n_{11})!}p_{110}^{n_{11}} \\ &\quad (p_{x0}-p_{110})^{x-n_{11}}\left(p_{y0}-p_{110}\right)^{y-n_{11}}\left(1-p_{x0}-p_{y0}+p_{110}\right)^{n-x-y+n_{11}}\Bigg) \times \\ &\quad \prod_{t=\tau+1}^{T}\sum_{n_{11}}\Bigg(\frac{n!}{n_{11}!(x-n_{11})!(y-n_{11})!(n-x-y+n_{11})!}p_{111}^{n_{11}}(p_{x1}-p_{111})^{x-n_{11}}\left(p_{y1}-p_{111}\right)^{y-n_{11}} \\ &\quad \left(1-p_{x1}-p_{y1}+p_{111}\right)^{n-x-y+n_{11}}\Bigg). \end{aligned} \tag{10}$$

The t in which the $L(t)$ has the maximum value is an estimation of the change point as Eq. (7).

3.2 Change point estimation using clustering approach

Generally, clustering is a statistical in which observations are divided according to their similarities into different groups called clusters. The most important issue in clustering is the validity of the clustering which means the correct defining of the clusters. Different criteria are defined to check the validity of clusters and the similarity index is one of them which is usually assessed by the distance between the centers of two clusters. In statistical process control, observations are divided into in-control and out-of-control states. These can be considered as two possible clusters, and observations are assigned to these

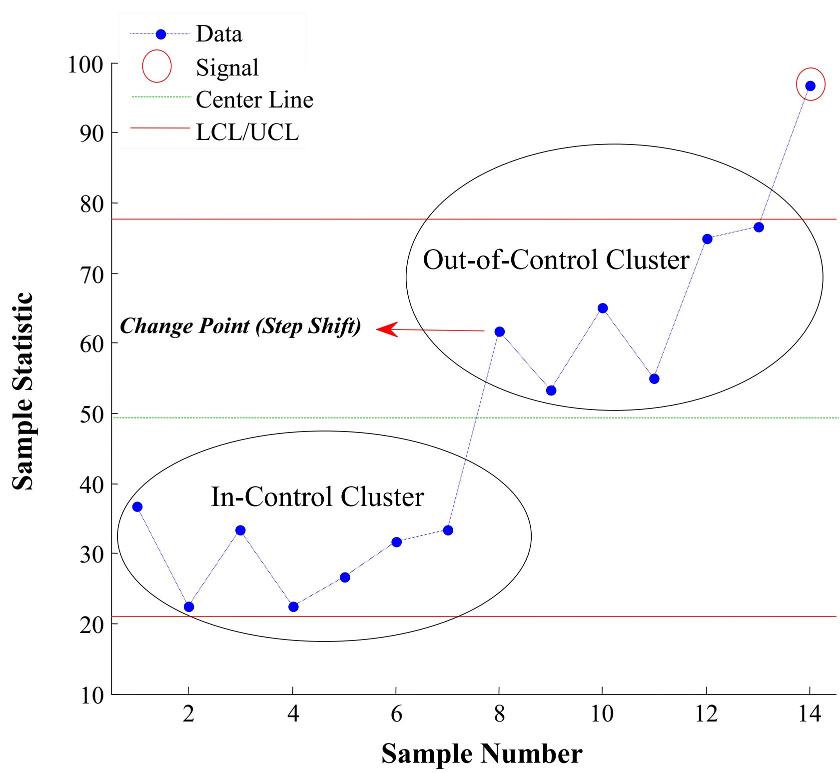

Fig. 1 Observations are divided to in-control and out-of-control clusters (Ghazanfari et al. [5])

clusters based on their similarity to in-control or out-of-control state (Fig. 1). The best two clusters can be defined based on the index and then the first observation in out-of-control cluster is known as the change point.

In other words, after an out-of-control alarm by a control chart, the samples are assigned to these two clusters according to their similarity to in-control or out-of-control parameters. The validity of clustering process is assessed by an index defined in Section 3.2.1. Since observations with several combinations can be classified into the clusters, the best combination is the one in which the validity index is optimum.

Assuming T, the time of shift detection, and the actual change point (τ), the first $\tau-1$ observations are located in the in-control cluster. Consequently, observations for $t=\tau$, $\tau+1$, …, T are placed in the out-of-control cluster. Since τ is unknown, it is possible that we assign the observations in to a wrong cluster (an observation which is actually in-control may be placed in the out-of-control cluster). In order to find value of τ, first, all possible combinations of observations' assignment to both the in- or out-of-control clusters are examined sequentially. Then, the point that maximizes the similarity index is introduced as the change point.

For example, assume that a control chart gives out-of-control signal at $T=14$. For estimating the change point, the similarity index should be assessed for all possible combinations. The procedure is shown schematically in Fig. 2.

Assume after evaluating index for all possible combinations, $D_k^2 = (\mathbf{p}_1 - \mathbf{p}_2)'\mathbf{S}^{-1}(\mathbf{p}_1 - \mathbf{p}_2)$, we conclude that the index reaches its optimum value with the combination of first 10 observations in the in-control cluster and the last 4 in the out-of-control cluster. Thus, the real change point is estimated in the observation 11.

3.2.1 Clustering validity index

The validity index can be defined based on the distance between centers of clusters. Here, the validity index is computed using Mahalanobis distance. Kurczynski [22] proposed a generalized Mahalanobis for discrete data as follows:

$$D_k^2 = (\mathbf{p}_1 - \mathbf{p}_2)'\mathbf{S}^{-1}(\mathbf{p}_1 - \mathbf{p}_2), \tag{11}$$

where k indicates class of attributes. Note that in the bivariate binomial distribution, there are two attributes each one have two classes (conforming and nonconforming). Here, $\mathbf{p}_i$ is vector of parameters p_{ij} for $i=1$, 2 and $j=1$, 2. Subscript i shows the in-control or out-of-control cluster (in-control cluster, $i=1$, and out-of-control cluster, $i=2$) and subscript j shows the attribute. p_{ij} is nonconformance probability of jth attribute in ith cluster. Also, $\mathbf{S}^{-1}$ is the inverse of the common sample covariance matrix.

Kurczynski [22] also showed that the Mahalanobis distance in Eq. (11) is equivalent to Eq. (12).

$$D_k^2 = \sum_{j=1}^{2}\sum_{k=1}^{2} d_{jk}^2 \Big/ p_{jk}, \tag{12}$$

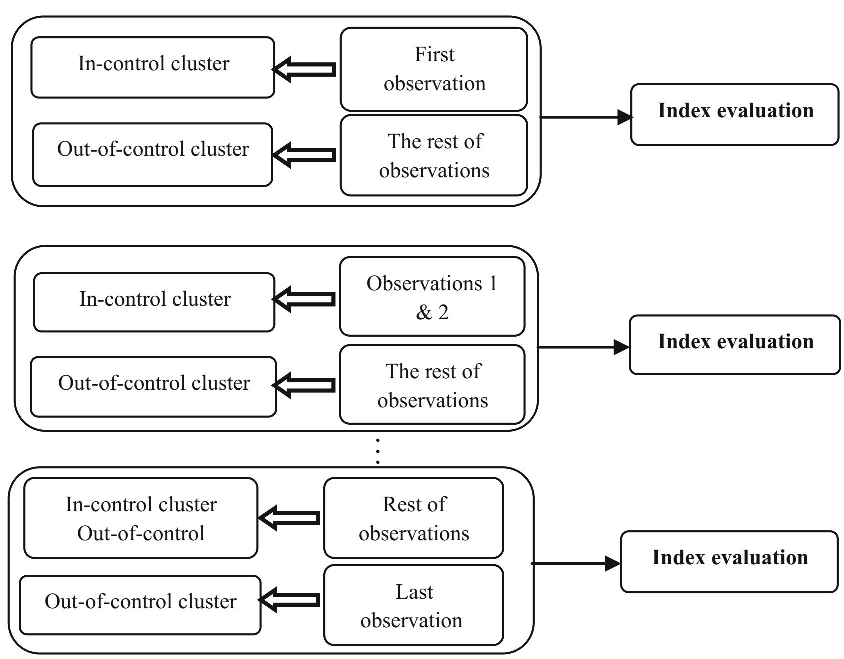

Fig. 2 The procedure of clustering method for change point estimation

where p_{jk} indicates the probability of kth class for jth attribute which is computed by $p_{jk} = \sum_{i=1}^{2} \left(n_{ij} p_{ijk} \right) / \sum_{i=1}^{2} n_{ij}$ in which n_{ij} is the sample size for the jth attribute in ith cluster. In addition, p_{ijk} is the probability of kth class for jth attribute in ith cluster. Also, d_{jk} is the difference between the probabilities of kth class of jth attribute in two clusters which is computed by using Eq. (13).

$$d_{jk} = p_{1jk} - p_{2jk}. \tag{13}$$

4 Performance evaluations

In this section, performance of the proposed methods are evaluated through numerical examples. It is assumed that the process mean is suddenly shifted. The size of this shift equals to $(\delta_1\sigma_1, \delta_2\sigma_2)$ and takes place at $t = 100$. Monte Carlo simulation is used to calculate the mean of estimated change points ($\overline{\widehat{\tau}}$), the standard error of the estimations $\left(\text{Stderror}\left(\overline{\widehat{\tau}}\right)\right)$ and the probability of estimating the change point with k units distance from the actual change point $(\text{P}(|\widehat{\tau}-\tau| \leq k))$. Obviously, $\overline{\widehat{\tau}}$ reflects the change point. Additionally, $\text{Stderror}\left(\overline{\widehat{\tau}}\right)$ and $\text{P}(|\widehat{\tau}-\tau| \leq k)$ are considered as a comparison basis of the two methods.

Suppose that samples are taken from a bivariate binomial distribution with known parameter. Firstly, 100 in control subgroups of random vectors are generated from the aforementioned distribution. If the X statistic of a subgroup computed by Eq. (1) exceeds the control limits, the subgroup is replaced with a new one. At $t=100$, the mean of process is shifted from μ_0 to μ_1 by $(\delta_1\sigma_1, \delta_2\sigma_2)$. In other words, the actual point in which the change has taken place is $t=100$. So, from $t=101$, the random vectors are generated from an out-of-control process until the Mnp control chart gives an out-of-control alarm. Then, $\overline{\widehat{\tau}}$ is calculated as the change point estimated by each of the proposed methods.

This procedure is repeated 5000 times considering different values of δ_1 and δ_2, $(\delta_1, \delta_2 = 0, 0.5, 1, 1.5, 2, 2.5, 3)$ for a bivariate binomial distribution (binomial (20,20;0.2,0.15;0.25))

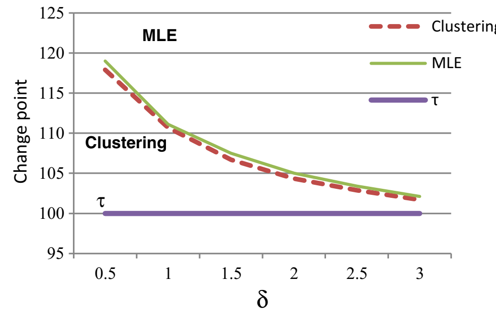

Fig. 3 The mean of change point estimations for clustering and MLE methods

and the results for $\overline{\widehat{\tau}}$, $\text{Stderror}\left(\overline{\widehat{\tau}}\right)$ and $\text{P}(|\widehat{\tau}-\tau| \leq k)$ obtained from each method are compared.

Table 2 denotes the results of simulations for $(\delta_1\sigma_1, 0\sigma_2)$ with the assumption that shift is merely caused by the first attribute. Note that E(T) is the average number of samples taken to get a signal plus 100 in-control samples.

As shown in the Table 2, both methods can estimate the change point effectively especially in large shifts. Also, it implies that the clustering method has better performance than the MLE based on the results in the Table 2 and Fig. 3.

Also, shift may occur in both attributes at the same time. Tables 3 and 4 show the result of simulations in which a shift with size of $\delta_2\sigma_2$ occurred in second attribute in addition to the first one.

The results of the Tables 3 and 4 illustrate that the clustering method performs better than the MLE method.

In addition to mean and variance of the estimations, the probability of estimating the change point with k units distance from the actual change point $\text{P}(|\widehat{\tau}-\tau| \leq k)$ is evaluated under different shift sizes and the results are summarized in Tables 5, 6, 7, and 8.

Tables 5 and 6 show the probabilities of the change point estimations with k unit distance from the actual change point in MLE and clustering methods. As the results show both methods reveal desirable performance in large shifts and estimate the actual change point well.

Tables 7 and 8 show that the precision of estimations is improved as the magnitude of the step shift increases.

Table 2 Values of E(T), $\overline{\widehat{\tau}}$ and Stderror($\overline{\widehat{\tau}}$) for MLE and clustering method for the shift of $(\delta_1\sigma_1, \delta_2\sigma_2)$ assuming $\delta_2 = 0$

δ_1		0.5	1	1.5	2	2.5	3
E(T)		121.14	113.3	109.1	106.3	104.3	103.4
Clustering	$\overline{\widehat{\tau}}$	117.91	110.7	106.7	104.33	102.9	101.7
	Stderror($\overline{\widehat{\tau}}$)	0.35	0.28	0.25	0.17	0.11	0.08
MLE	$\overline{\widehat{\tau}}$	119.8	111.7	107.8	105	103.2	102.1
	Stderror($\overline{\widehat{\tau}}$)	0.36	0.28	0.25	0.18	0.12	0.08

Table 3 Values of $\mathrm{E}(T)$, $\overline{\hat{\tau}}$ and Stderror($\overline{\hat{\tau}}$) for MLE and clustering method for the shift of $(\delta_1\sigma_1, \delta_2\sigma_2)$ assuming $\delta_2 = 1$

δ_1		0	0.5	1	1.5	2	2.5	3
E(T)		109.7	107	105.1	104	103.1	102.5	102
Clustering	$\overline{\hat{\tau}}$	108	103.9	102.4	101.5	101.1	100.7	100.22
	Stderror($\overline{\hat{\tau}}$)	0.28	0.27	0.23	0.19	0.16	0.11	0.07
MLE	$\overline{\hat{\tau}}$	108.5	105.5	103.5	102.4	101.5	101.2	100.7
	Stderror ($\overline{\hat{\tau}}$)	0.29	0.28	0.23	0.19	0.15	0.11	0.07

Table 4 Values of $\mathrm{E}(T)$, $\overline{\hat{\tau}}$ and Standard derror($\overline{\hat{\tau}}$) for MLE and clustering method for the shift of $(\delta_1\sigma_1, \delta_2\sigma_2)$ assuming $\delta_2 = 2$

δ_1		0	0.5	1	1.5	2	2.5	3
E(T)		103.9	103.1	102.5	101.8	101.6	101.4	101.3
Clustering	$\overline{\hat{\tau}}$	102.5	100.5	100.3	100.3	100.2	100.2	100.1
	Stderror($\overline{\hat{\tau}}$)	0.22	0.21	0.19	0.17	0.14	0.11	0.07
MLE	$\overline{\hat{\tau}}$	102.8	101.9	101.4	100.75	100.6	100.4	100.2
	Stderror($\overline{\hat{\tau}}$)	0.21	0.28	0.19	0.18	0.15	0.11	0.07

Basically, it can be concluded that both methods perform well especially in large shifts. Moreover, the clustering method estimates the actual change point better than the MLE method. Also, clustering due to simplicity is a more applicable method than the complicated MLE method.

5 Conclusions and future researches

In this paper, two methods including MLE and clustering proposed for change point estimation in a process with bivariate binomial distribution. The performance of these methods are evaluated and compared by Monte Carlo simulation. The results show that both estimators can find the real change point effectively especially in large shifts. However, clustering method performs better than the MLE method. Also due to simplicity of the proposed clustering method, it seems that this method is more applicable than the MLE method for change point estimations in real world processes. As a future research, the performance of the proposed estimator can be compared with the competing estimators such as artificial neural network or genetic algorithm. The proposed estimators of this

Table 5 Probability of estimating the change point with k unit distance from the actual change point in clustering method assuming shift size of $(\delta_1\sigma_1, \delta_2\sigma_2)$ and δ_2=0

δ_1 Probability	0.5	1	1.5	2	2.5	3
$P(\lvert\hat{\tau}-\tau\rvert = 0)$	0.04	0.07	0.12	0.17	0.25	0.36
$P(\lvert\hat{\tau}-\tau\rvert \le 1)$	0.07	0.17	0.21	0.32	0.45	0.56
$P(\lvert\hat{\tau}-\tau\rvert \le 2)$	0.10	0.22	0.32	0.43	0.55	0.70
$P(\lvert\hat{\tau}-\tau\rvert \le 3)$	0.16	0.29	0.40	0.53	0.66	0.78
$P(\lvert\hat{\tau}-\tau\rvert \le 4)$	0.22	0.33	0.47	0.61	0.75	0.87
$P(\lvert\hat{\tau}-\tau\rvert \le 5)$	0.24	0.38	0.53	0.67	0.78	0.91
$P(\lvert\hat{\tau}-\tau\rvert \le 6)$	0.29	0.39	0.58	0.72	0.84	0.94
$P(\lvert\hat{\tau}-\tau\rvert \le 7)$	0.3	0.46	0.62	0.77	0.88	0.95
$P(\lvert\hat{\tau}-\tau\rvert \le 8)$	0.34	0.49	0.68	0.80	0.89	0.96
$P(\lvert\hat{\tau}-\tau\rvert \le 9)$	0.37	0.54	0.71	0.86	0.93	0.98
$P(\lvert\hat{\tau}-\tau\rvert \le 10)$	0.42	0.57	0.76	0.88	0.97	0.99
$P(\lvert\hat{\tau}-\tau\rvert \le 15)$	0.53	0.73	0.86	0.94	0.99	

Table 6 Probability of estimating the change point with k unit distance from the actual change point in the MLE method assuming shift size of $(\delta_1\sigma_1, \delta_2\sigma_2)$ and $\delta_2 = 0$

δ_1 probability	0.5	1	1.5	2	2.5	3
$P(\lvert\hat{\tau}-\tau\rvert = 0)$	0.03	0.07	0.11	0.19	0.24	0.32
$P(\lvert\hat{\tau}-\tau\rvert \le 1)$	0.09	0.15	0.22	0.31	0.44	0.52
$P(\lvert\hat{\tau}-\tau\rvert \le 2)$	0.14	0.22	0.31	0.42	0.55	0.66
$P(\lvert\hat{\tau}-\tau\rvert \le 3)$	0.19	0.29	0.40	0.52	0.63	0.78
$P(\lvert\hat{\tau}-\tau\rvert \le 4)$	0.22	0.34	0.46	0.60	0.73	0.85
$P(\lvert\hat{\tau}-\tau\rvert \le 5)$	0.27	0.41	0.54	0.66	0.79	0.90
$P(\lvert\hat{\tau}-\tau\rvert \le 6)$	0.29	0.43	0.57	0.70	0.86	0.93
$P(\lvert\hat{\tau}-\tau\rvert \le 7)$	0.34	0.46	0.66	0.87	0.88	0.95
$P(\lvert\hat{\tau}-\tau\rvert \le 8)$	0.35	0.50	0.66	0.81	0.91	0.96
$P(\lvert\hat{\tau}-\tau\rvert \le 9)$	0.42	0.54	0.72	0.84	0.93	0.97
$P(\lvert\hat{\tau}-\tau\rvert \le 10)$	0.43	0.61	0.74	0.88	0.95	0.99
$P(\lvert\hat{\tau}-\tau\rvert \le 15)$	0.55	0.73	0.86	0.95	0.98	

Table 7 Probability of estimating the change point with k unit distance from the actual change point in clustering method assuming shift size of $(\delta_1\sigma_1, \delta_2\sigma_2)$ and $\delta_2 = 1.5$

δ_1 Probability	0	0.5	1	1.5	2	2.5	3
$P(\lvert\hat{\tau}-\tau\rvert = 0)$	0.18	0.25	0.33	0.45	0.53	0.60	0.68
$P(\lvert\hat{\tau}-\tau\rvert \le 1)$	0.33	0.43	0.57	0.66	0.76	0.84	0.91
$P(\lvert\hat{\tau}-\tau\rvert \le 2)$	0.43	0.56	0.71	0.81	0.89	0.94	0.97
$P(\lvert\hat{\tau}-\tau\rvert \le 3)$	0.54	0.69	0.81	0.89	0.95	0.97	0.98
$P(\lvert\hat{\tau}-\tau\rvert \le 4)$	0.61	0.76	0.87	0.93	0.96	0.98	0.99
$P(\lvert\hat{\tau}-\tau\rvert \le 5)$	0.67	0.80	0.91	0.96	0.98	0.99	
$P(\lvert\hat{\tau}-\tau\rvert \le 6)$	0.73	0.87	0.94	0.97	0.99		
$P(\lvert\hat{\tau}-\tau\rvert \le 7)$	0.76	0.89	0.95	0.98			
$P(\lvert\hat{\tau}-\tau\rvert \le 8)$	0.82	0.91	0.96	0.99			
$P(\lvert\hat{\tau}-\tau\rvert \le 9)$	0.84	0.93	0.98				
$P(\lvert\hat{\tau}-\tau\rvert \le 10)$	0.86	0.96	0.99				
$P(\lvert\hat{\tau}-\tau\rvert \le 15)$	0.95	0.99					

Table 8 Probability of estimating the change point with k unit distance from the actual change point in the MLE method assuming shift size of $(\delta_1\sigma_1, \delta_2\sigma_2)$ and δ_2= 1.5

δ_1 Probability	0	0.5	1	1.5	2	2.5	3
$P(\lvert\hat{\tau}-\tau\rvert = 0)$	0.17	0.26	0.34	0.43	0.53	0.59	0.68
$P(\lvert\hat{\tau}-\tau\rvert \le 1)$	0.32	0.45	0.57	0.68	0.76	0.85	0.88
$P(\lvert\hat{\tau}-\tau\rvert \le 2)$	0.44	0.58	0.70	0.81	0.88	0.93	0.96
$P(\lvert\hat{\tau}-\tau\rvert \le 3)$	0.52	0.70	0.81	0.88	0.94	0.97	0.99
$P(\lvert\hat{\tau}-\tau\rvert \le 4)$	0.61	0.76	0.87	0.93	0.97	0.98	
$P(\lvert\hat{\tau}-\tau\rvert \le 5)$	0.69	0.82	0.92	0.97	0.98	0.99	
$P(\lvert\hat{\tau}-\tau\rvert \le 6)$	0.73	0.88	0.95	0.98	0.99		
$P(\lvert\hat{\tau}-\tau\rvert \le 7)$	0.78	0.90	0.96	0.98			
$P(\lvert\hat{\tau}-\tau\rvert \le 8)$	0.81	0.93	0.97	0.99			
$P(\lvert\hat{\tau}-\tau\rvert \le 9)$	0.84	0.95	0.98				
$P(\lvert\hat{\tau}-\tau\rvert \le 10)$	0.88	0.96	0.99				
$P(\lvert\hat{\tau}-\tau\rvert \le 15)$	0.96	0.99					

paper can also be extended to multivariate binomial distribution as well.

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